Nonstandard Analysis and Combinatorial Number Theory

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Day Two: Basic Methods

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OUTLINE:

1. Properties and Principles
2. Loeb Space Construction
3. Application to Finance

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$$\forall x \in \mathscr{P}([0,1])(x \text{ has a least upper bound in } [0,1]). \qquad (1)$$

Can we conclude by the transfer principle that the sentence

$$\forall x \in \mathscr{P}({}^*[0,1])(x \text{ has a least upper bound in } {}^*[0,1])$$

is true in ${}^*\mathcal{V}$? Of course, ${}^*\mathcal{R}$ in ${}^*\mathcal{V}$ should not satisfy the completeness property because there is no least upper bound of all infinitesimals. Does this cause inconsistency? To clarify the issue we should pay attention to the difference between internal sets and external sets.

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Let $A \in \mathcal{V}$ be a set with rank $\leq \mathfrak{n}$. A subset A_0 of A is finite iff there is a bijection in $\mathcal{V}$ between A_0 and $[n]$ for some $n \in \mathbb{N}$. We denote $|A_0| = n$ for saying that A_0 has a cardinality n. The cardinality function $|\cdot|$ can be extend to a function $^*|\cdot|$ from all *finite subsets of $^*A$ to $^*\mathbb{N}$. So, $^*|A_1| = n$ iff there is a bijection in $^*\mathcal{V}$ between A_1 and $[n]$. For notational convenience, we omit * from $^*|\cdot|$. A set A_1 is called a hyperfinite set if $|A_1|$ is a hyperfinite integer.

Definition (2.1)

Every element or set of the form $^*a$ for some $a \in \mathcal{V}$ is called standard and every element or set $a \in {}^*\mathcal{V}$ is called internal. If an element or a set is not in $^*\mathcal{V}$, we call it external.

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Example (2.2)

1. *Every $r \in \mathbb{R}$ is standard, and $^*\mathbb{N}$, $^*\mathbb{R}$ are standard.*
2. *For each hyperfinite integer N the sets $[N]$ and $[-N, N] \cap {}^*\mathbb{R}$ are internal but not standard.*
3. *The sets $\mathbb{N}$ and $\mathbb{R}$ are external subsets of $^*\mathbb{R}$.*

For Part 2 above let $N - 1 = [g]$ where $g : \mathbb{N} \to \mathbb{N}$ and $\{n \in \mathbb{N} \mid g(n) > m\} \in \mathcal{F}$ for each $m \in \mathbb{N}$.

If $^*a = {}^*\mathbb{N} \cap [0, [g]] \in {}^*\mathcal{V}$ is standard, then $^*a$ being bounded above in $^*\mathbb{N}$ implies a being bounded above in $\mathbb{N}$ by the transfer property. This means that a is a finite subset of $\mathbb{N}$. So, we have $^*a = a$ which is a finite set contradicting that $^*a$ is a hyperfinite set. Hence, $^*\mathbb{N} \cap [N]$ is internal but not standard. By a similar reason, the set $^*\mathbb{R} \cap [-N, N]$ is internal but not standard.

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Note that the statement $\mathcal{V} \models \varphi$ for φ being in (1) is transferred to ${}^*\mathcal{V}$ to become

$${}^*\mathcal{V} \models \forall x \in {}^*\mathscr{P}({}^*[0,1])(x \text{ has a least upper bound in } {}^*[0,1]).$$

The reader should notice the difference between ${}^*\mathscr{P}({}^*[0,1])$ and $\mathscr{P}({}^*[0,1])$. The former is the collection of all internal subsets of ${}^*[0,1]$ and the latter is the collection of all subsets (internal or external) of ${}^*[0,1]$. So, in ${}^*\mathcal{V}$ every internal subset of ${}^*[0,1]$ has a least upper bound. Therefore, the set of all infinitesimals in ${}^*\mathbb{R}$ is not an internal set.

For Part 3 above, since every bounded subset of $\mathbb{N}$ is finite and has a maximal element in $\mathbb{N}$, by the transfer principle, every bounded internal subset of ${}^*\mathbb{N}$ is finite or hyperfinite and has a maximal element. But $\mathbb{N}$ as a subset of ${}^*\mathbb{N}$ does not have a maximal element. Therefore, $\mathbb{N}$ is not internal in ${}^*\mathbb{N}$. By a similar reason, $\mathbb{R}$ is not an internal subset of ${}^*\mathbb{R}$.

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Proposition (2.3, Definability of Internal Sets)

Let $A \in {}^*\mathcal{V}$ be an internal set with rank$(A) \leq \mathfrak{n}$ and $\varphi(\overline{x}, \overline{b})$ be a formula with parameters $\overline{b}$ in ${}^*\mathcal{V}$ where $\overline{x}$ is an m-tuple of variables. Then

$$\{\overline{a} \in A^m \mid {}^*\mathcal{V} \models \varphi(\overline{a}, \overline{b})\} \tag{2}$$

is again an internal subset of A^m.

Proof. Let $A = [f]$ and $\overline{b} = \overline{[g]}$. Define a function $h : \mathbb{N} \to \mathcal{V}$ by letting

$$h(n) := \{\overline{a} \in f(n)^m \mid \mathcal{V} \models \varphi(\overline{a}, \overline{g(n)})\}$$

for each $n \in \mathbb{N}$. Let $B = [h]$. Then B is an internal subset of A^m. The proposition follows because

$$\overline{[p]} \in B \text{ iff } \{n \in \mathbb{N} \mid \overline{p(n)} \in h(n)\} \in \mathcal{F}$$
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A nonempty set $U \subseteq {}^*\mathbb{N}$ is an initial segment of ${}^*\mathbb{N}$ if $n \in U$ and $m < n$ imply $m \in U$ for any $m, n \in {}^*\mathbb{N}$. For example, $\mathbb{N}$ is an external initial segment of ${}^*\mathbb{N}$.

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Proposition (2.4, Overspill and Underspill Principle)

Let U be an external initial segment of ${}^*\mathbb{N}$ and A be an internal subset of ${}^*\mathbb{N}$.

1. *If $A \cap U$ is unbounded above in U, then $A \setminus U \neq \emptyset$;*
2. *If $A \setminus U$ is unbounded below in ${}^*\mathbb{N} \setminus U$, then $A \cap U \neq \emptyset$.*

Proof. Part 1: Suppose $A \setminus U = \emptyset$. Then

$$U = \{x \in {}^*\mathbb{N} \mid \exists a \in A(x \leq a)\}$$

is internal by Proposition 2.3 which contradicts the assumption that U is external. The proof of Part 2 is similar. □

The overspill and underspill principles are frequently used tools in nonstandard analysis.

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is internal by Proposition 2.3 which contradicts the assumption that U is external. The proof of Part 2 is similar. □

The overspill and underspill principles are frequently used tools in nonstandard analysis.

Proposition (2.4, Overspill and Underspill Principle)

Let U be an external initial segment of ${}^*\mathbb{N}$ and A be an internal subset of ${}^*\mathbb{N}$.

1. *If $A \cap U$ is unbounded above in U, then $A \setminus U \neq \emptyset$;*
2. *If $A \setminus U$ is unbounded below in ${}^*\mathbb{N} \setminus U$, then $A \cap U \neq \emptyset$.*

Proof: Part 1: Suppose $A \setminus U = \emptyset$. Then

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Proposition (2.5, Countable Saturation)

Let A be an infinite internal set in ${}^*\mathcal{V}$ with rank $\leq \mathfrak{n}$ and $A \supseteq B_0 \supseteq B_1 \supseteq \cdots$ be a nested sequence of nonempty internal sets. Then,

$$\bigcap_{m \in \mathbb{N}} B_m \neq \emptyset.$$

Proof. Let $B_m = [b_m]$ for some $b_m \in \mathcal{V}^{\mathbb{N}}$ and choose an $[f_m] \in [b_m]$. For each $m \in \mathbb{N}$ let

$$U_m := \{n \in \mathbb{N} \mid n > m,\ f_m(n) \in b_m(n), \text{ and } b_0(n) \supseteq b_1(n) \supseteq \cdots \supseteq b_m(n)\}.$$

Then $U_m \in \mathcal{F}$. For each $n \in \mathbb{N}$, let $m_n := \max\{m \in \mathbb{N} \mid n \in U_m\}$. Note that m_n exists because $\bigcap_{m \in \mathbb{N}} U_m = \emptyset$. Note also that $n \in U_{m_n}$.

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Let $f \in \mathcal{V}^{\mathbb{N}}$ be a function such that $f(n) = f_{m_n}(n)$ for every $n \in \mathbb{N}$. It suffices to show that $[f] \in [b_m]$ for every $m \in \mathbb{N}$.

Given $m \in \mathbb{N}$, let $U := \{n \in \mathbb{N} \mid f(n) \in b_m(n)\}$. For each $n \in U_m$, we have $m \leq m_n$ by the maximality of m_n.

Since $n \in U_{m_n}$, we have $f(n) = f_{m_n}(n) \in b_{m_n}(n) \subseteq b_m(n)$. Hence, $n \in U$ which means $U_m \subseteq U$.

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Countable saturation was used first by W. A. J. Luxemburg in 1969. It is a key property in the development of Loeb measure.

In Proposition 2.3 and Proposition 2.5 the set A is assumed to have rank $\leq \mathfrak{n}$ because some collection of subsets of A are mentioned which may have rank greater than $\mathfrak{n}$. Since the elements with rank higher than $\mathfrak{n}$ are still in $\mathcal{V}$ as long as the rank is $\leq 2\mathfrak{n}$.

If the set A has a rank $2\mathfrak{n}$, then some objects needed will be outside of $\mathcal{V}$.

Since all mathematical objects in our applications will have a rank $\leq \mathfrak{n}$ the restriction $\text{rank}(A) \leq \mathfrak{n}$ will not cause any problem.

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Proposition 2.5 is still true if the sequence B_m is assumed to satisfy the finite intersection property, i.e., the intersection of any finite collection of B_m's is nonempty, instead of the sequence being nested.

Proposition 2.5 is also true if $\mathcal{F}$ is a non-principal ultrafilter on any infinite set X as long as it is countably incomplete.

For any infinite cardinal κ, there exist ultrafilters $\mathcal{F}$ such that the ultrapower of $\mathcal{V}$ modulo $\mathcal{F}$ satisfies κ-saturation property, i.e., any collection of less than κ many internal subsets of an internal set satisfying finite intersection property has a nonempty intersection.

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The next two corollaries are trivial.

Corollary (2.6)

Every internal set A in ${}^*\mathcal{V}$ is either finite or uncountable.

Corollary (2.7)

Let U be an infinite initial segment of ${}^*\mathbb{N}$. Let $\{x_n \in U \mid n \in \mathbb{N}\}$ be increasing and $\{y_n \in {}^*\mathbb{N} \setminus U \mid n \in \mathbb{N}\}$ be decreasing. Then either $\{x_n \in U \mid n \in \mathbb{N}\}$ is bounded above by some $z \in U$ or $\{y_n \in {}^*\mathbb{N} \setminus U \mid n \in \mathbb{N}\}$ is bounded below by some $z \in {}^*\mathbb{N} \setminus U$.

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Corollary (2.8)

Let $A \in {}^*\mathcal{V}$ and $s : \mathbb{N} \to A$ be an external sequence. There exists an internal function $S : {}^*\mathbb{N} \to A$ such that $S \restriction \mathbb{N} = s$.

Proof: For each $m \in \mathbb{N}$ let

$$\mathcal{S}_m := \{t \in {}^*\mathcal{V} \mid t : {}^*\mathbb{N} \to A\,(t(i) = s(i) \text{ for } i \in [m+1]).$$

Note that $\mathcal{S}_m \in {}^*\mathscr{P}(A^{{}^*\mathbb{N}} \cap {}^*\mathcal{V})$ is nonempty because it contains at least an internal function s' such that $s'(i) = s(i)$ for $i \in [m+1]$ and $s'(i) = s(0)$ for any $i \in {}^*\mathbb{N} \setminus [m]$. It is easy to see that $\mathcal{S}_m \supseteq \mathcal{S}_{m+1}$.

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Remark (2.9)

Note that if $s : \mathbb{N} \to A$ *is an injection, we cannot require that* $S : {}^*\mathbb{N} \to A$ *be an injection in Corollary 2.8. However, if*

$$B := \{m \in {}^*\mathbb{N} \mid S \restriction [m+1] \text{ is an injection}\},$$

then B *is internal and upper unbounded in* $\mathbb{N}$.

By the overspill principle the set B *contains some hyperfinite integer* N. *Hence,* $S \restriction [N+1]$ *is an injection from* $[N+1]$ *to* A.

For example, a strictly increasing sequence $\{r_i \mid i \in \mathbb{N}\}$ *in some interval* $[a, b] \subseteq {}^*\mathbb{R}$ *may not be extended to an internal strictly increasing sequence* $\{r_i \mid i \in {}^*\mathbb{N}\}$ *in* $[a, b]$. *Instead, it can be extended to a hyperfinite strictly increasing sequence* $\{r_i \mid 0 \leq i \leq N\}$ *for some hyperfinite integer* N.

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In this course we introduce only Loeb probability space generated by an internal normalized counting measure on a hyperfinite set.

Definition (2.10)

Let Ω be a hyperfinite set in ${}^*\mathcal{V}$ and $\Sigma_0 := {}^*\mathscr{P}(\Omega)$ be the set of all internal subsets of Ω. Clearly, each $A \in \Sigma_0$ is a finite or hyperfinite set. For $A \in \Sigma_0$ define

$$\delta(A) := \frac{|A|}{|\Omega|} \in {}^*[0,1] \text{ and } \mu_\Omega(A) := st(\delta(A)) \in [0,1].$$

Then, $(\Omega; \Sigma_0, \delta)$ is called a normalized counting measure space, and $(\Omega; \Sigma_0, \mu_\Omega)$ is called a standardized normalized counting measure space.

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Let $(\Omega; \Sigma, \mu_\Omega)$ be a Loeb space defined in Definition 2.2. Then,

(1) $\Sigma_0 \subseteq \Sigma$;
Part 1 is true because of the definition of lower and upper measure.

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(5) *Let $X \subseteq \Omega$. Then, $X \in \Sigma$ iff X has squeezing sandwich sequences of internal sets A_i and B_i for $i \in \mathbb{N}$, i.e., (sandwich)*

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(6.1): Since $A_m \cup A'_m$ and $B_m \cup B'_m$ are sandwich sequences of $X \cup Y$ and

$$(B_m \cup B'_m) \setminus (A_m \cup A'_m) \subseteq (B_m \setminus A_m) \cup (B'_m \setminus A'_m),$$

we have that

$$\lim_{m\to\infty} \mu_\Omega((B_m \cup B'_m) \setminus (A_m \cup A'_m))$$
$$\leq \lim_{m\to\infty} \mu_\Omega(B_m \setminus A_m) + \lim_{m\to\infty} \mu_\Omega(B'_m \setminus A'_m) = 0,$$

which implies $X \cup Y \in \Sigma$ by Part 5 and hence,

$$|\mu_\Omega(X \cup Y) = \lim_{m\to\infty} \mu_\Omega(B_m \cup B'_m)$$
$$\leq \lim_{m\to\infty} \mu_\Omega(B_m) + \lim_{m\to\infty} \mu_\Omega(B'_m) = \mu_\Omega(X) + \mu_\Omega(Y).$$

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(6.1): Since $A_m \cup A'_m$ and $B_m \cup B'_m$ are sandwich sequences of $X \cup Y$ and

$$(B_m \cup B'_m) \setminus (A_m \cup A'_m) \subseteq (B_m \setminus A_m) \cup (B'_m \setminus A'_m),$$

we have that

$$\lim_{m\to\infty} \mu_\Omega((B_m \cup B'_m) \setminus (A_m \cup A'_m)) \leq \lim_{m\to\infty} \mu_\Omega(B_m \setminus A_m) + \lim_{m\to\infty} \mu_\Omega(B'_m \setminus A'_m) = 0,$$

which implies $X \cup Y \in \Sigma$ by Part 5 and hence,

$$|\mu_\Omega(X \cup Y) = \lim_{m\to\infty} \mu_\Omega(B_m \cup B'_m) \leq \lim_{m\to\infty} \mu_\Omega(B_m) + \lim_{m\to\infty} \mu_\Omega(B'_m) = \mu_\Omega(X) + \mu_\Omega(Y).$$

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(6.2): Note that $A_m \setminus B'_m \subseteq X \setminus Y \subseteq B_m \setminus A'_m$, *which mean* $A_m \setminus B'_m$ *and* $B_m \setminus A'_m$ *are sandwich sequences for* $X \setminus Y$.

Since $(B_m \setminus A'_m) \setminus (A_m \setminus B'_m) \subseteq (B_m \setminus A_m) \cup (B'_m \setminus A'_m)$, *we have that* $A_m \setminus B'_m$ *and* $B_m \setminus A'_m$ *are squeezing.*

So, $X \setminus Y \in \Sigma$ *and* $\mu_\Omega(X \setminus Y) = \lim_{m\to\infty} \mu_\Omega(B_m \setminus A'_m) = \lim_{m\to\infty} \mu_\Omega(B_m) - \lim_{m\to\infty} \mu_\Omega(A'_m) = \mu_\Omega(X) - \mu_\Omega(Y)$.

In particular, we have $X^c \in \Sigma$, *where* $X^c := \Omega \setminus X$, *and* $\mu_\Omega(X^c) = 1 - \mu_\Omega(X)$.

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(6.3): *If* $X \cap Y = \emptyset$, *then* $Y \subseteq X^c$. *Hence,* $X \cup Y = (X^c \cap Y^c)^c = (X^c \setminus Y)^c$ *and* $\mu_\Omega(X \cup Y) = 1 - (\mu(X^c \setminus Y)) = 1 - (\mu_\Omega(X^c) - \mu_\Omega(Y)) = 1 - (1 - \mu_\Omega(X) - \mu_\Omega(Y)) = \mu_\Omega(X) + \mu_\Omega(Y)$.

(6.4): $X \setminus Y = X \cap Y^c = (X^c \cup Y)^c \in \Sigma$.

(7) *If* $X \in \Sigma$, *then there exists* $K \in \Sigma_0$ *such that* $\mu_\Omega(X \Delta K) = 0$, *where* $X \Delta K := (X \setminus K) \cup (K \setminus X)$;

Part 7: Let A_m *and* B_m *be a squeezing sandwich sequences for* X. *Let* $\mathcal{K}_m = \{K \in \Sigma_0 \mid A_m \subseteq K \subseteq B_m\}$. *Then,* $\mathcal{K}_m$ *is nonempty, internal, and* $\mathcal{K}_{m+1} \subseteq \mathcal{K}_m$. *By Proposition 2.5 there is a* $K \in \bigcap_{m \in \mathbb{N}} \mathcal{K}_m$. *Clearly,* A_m, B_m *are squeezing sandwich sequences for* K. *Since* $X \Delta K \subseteq B_m \setminus A_m$, *we have that* $\mu_\Omega(X \Delta K) \leq \mu_\Omega(B_m \setminus A_m) \to 0$. *So,* $\mu_\Omega(X \Delta K) = 0$.

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(8) *If $X_i \in \Sigma$ for $i \in \mathbb{N}$ is a pairwise disjoint sequence, then*

$$\mu_\Omega\left(\bigcup_{i\in\mathbb{N}} X_i\right) = \sum_{i\in\mathbb{N}} \mu_\Omega(X_i);$$

Part 8: By passing to subsequences we can find squeezing sandwich sequences $A_m^{(i)}, B_m^{(i)}$ for each X_i such that

$$\max\{\mu_\Omega(B_m^{(i)} \setminus X_i), \mu_\Omega(X_i \setminus A_m^{(i)})\} \le \mu_\Omega(B_m^{(i)} \setminus A_m^{(i)}) < 1/2^i m.$$

Note that $A_m^{(i)}$ for $i = 1, 2, \ldots$ are pairwise disjoint.

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For each $m \in \mathbb{N}$ we can find a hyperfinite integer N_m such that the sequences $\{A_m^{(i)}, B_m^{(i)} \mid i \in \mathbb{N}\}$ can be extended to internal sequences $\{A_m^{(i)}, B_m^{(i)} \mid 1 \leq i \leq N_m\}$ such that $\delta(B_m^{(i)} \setminus A_m^{(i)}) < 1/2^i m$ for $0 \leq i \leq N_m$. By Corollary 2.7 there is a hyperfinite integer $N \leq N_m$ for every $m \in \mathbb{N}$. So, for any $m \in \mathbb{N}$ and $0 \leq i \leq N$ we have $\delta(B_m^{(i)} \setminus A_m^{(i)}) < 1/2^i m$. For each $m \in \mathbb{N}$ let

$$B_m := \bigcup_{i=1}^{N} B_m^{(i)} \text{ and } A_m := \bigcup_{i=1}^{m} A_m^{(i)}.$$

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Proposition (2.12 with proof)

Clearly, A_m, B_m *are sandwich sequences of internal sets for* $X := \bigcup_{i\in\mathbb{N}} X_i$. *It suffices to show that the sequences are also squeezing.*

Since

$$\sum_{i=1}^{m} \mu_\Omega(X_i) = \mu_\Omega\left(\bigcup_{i=1}^{m} X_i\right) \leq 1$$

by Part 6, we have that $\lim_{m\to\infty} T_m = 0$ *where*

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Given any $m' > m$ in $\mathbb{N}$, we have

$$\sum_{i=m+1}^{m'} \delta(B_m^{(i)}) \leq \frac{1}{m} + \sum_{i=m+1}^{m'} \mu_\Omega(B_m^{(i)} \setminus X_i) + \sum_{i=m+1}^{m'} \mu_\Omega(X_i)$$

$$\leq \frac{1}{m} + \frac{1}{m} \sum_{i=m+1}^{m'} \frac{1}{2^i} + T_m \leq \frac{1}{m} + \frac{1}{m2^m} + T_m \leq \frac{2}{m} + T_m.$$

By extending m' to hyperfinite we can assume that

$$\delta\left(\bigcup_{i=m+1}^{N} B_m^{(i)}\right) \leq \sum_{i=m+1}^{N} \delta(B_m^{(i)}) \leq \frac{2}{m} + T_m.$$

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$$
\begin{aligned}
\text{So, } & \mu_\Omega(B_m \setminus A_m) \\
& \leq \mu_\Omega\left(\bigcup_{i=1}^{m} B_m^{(i)} \setminus \bigcup_{i=1}^{m} A_m^{(i)}\right) + \mu_\Omega\left(\bigcup_{i=m+1}^{N} B_m^{(i)}\right) \\
& \leq \mu_\Omega\left(\bigcup_{i=1}^{m} (B_m^{(i)} \setminus A_m^{(i)})\right) + \frac{3}{m} + T_m \\
& \leq \sum_{i=1}^{m} \frac{1}{2^i m} + \frac{3}{m} + T_m \leq \frac{4}{m} + T_m \to 0
\end{aligned}
$$

as $m \to \infty$. *Therefore,* A_m, B_m *are squeezing for* X *which implies* $X \in \Sigma$.

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\textit{So, } \; & \mu_\Omega(B_m \setminus A_m) \\
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Proposition (2.12 with proof)

Note that

$$\begin{aligned}\mu_\Omega(X) &= \sum_{i=1}^{m} \mu_\Omega(X_i) + \mu_\Omega\left(\bigcup_{i=m+1}^{\infty} X_i\right) \\ &\leq \sum_{i=1}^{m} \mu_\Omega(X_i) + \frac{1}{m} + \delta\left(\bigcup_{i=m+1}^{N} B_m^{(i)}\right) \\ &\leq \sum_{i=1}^{m} \mu_\Omega(X_i) + \frac{3}{m} + T_m \to \sum_{i=1}^{\infty} \mu_\Omega(X_i)\end{aligned}$$

as $m \to \infty$, *and*

Proposition (2.12 with proof)

$$\mu_\Omega(X) = \lim_{m\to\infty} \mu_\Omega(A_m) = \lim_{m\to\infty} \sum_{i=1}^{m} \mu_\Omega(A_m^{(i)})$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m} \left(\mu_\Omega(X_i) - \mu_\Omega(X_i \setminus A_m^{(i)})\right)$$

$$\geq \lim_{m\to\infty} \sum_{i=1}^{m} \left(\mu_\Omega(X_i) - \frac{1}{2^i m}\right)$$

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We conclude that $\mu_\Omega(X) = \sum_{i\in\mathbb{N}} \mu_\Omega(X_i)$.

Proposition (2.12 with proof)

$$\mu_\Omega(X) = \lim_{m\to\infty} \mu_\Omega(A_m) = \lim_{m\to\infty} \sum_{i=1}^{m} \mu_\Omega(A_m^{(i)})$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m} \left(\mu_\Omega(X_i) - \mu_\Omega(X_i \setminus A_m^{(i)})\right)$$

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$$= \lim_{m\to\infty} \sum_{i=1}^{m} \mu_\Omega(X_i) - \lim_{m\to\infty} \sum_{i=1}^{m} \frac{1}{2^i m} = \sum_{i=1}^{\infty} \mu_\Omega(X_i).$$

We conclude that $\mu_\Omega(X) = \sum_{i\in\mathbb{N}} \mu_\Omega(X_i)$.

Proposition (2.12 with proof)

(9) *Σ is a σ-algebra and $(\Omega; \Sigma, \mu_\Omega)$ is an atomless, complete, countably additive probability space in the standard sense.*

Part 9: Σ is a σ-algebra by Part 6 and 8.
$(\Omega; \Sigma, \mu_\Omega)$ is complete by Part 3, and
countably additive by Part 8.

If $X \in \Sigma$ with $\mu_\Omega(X) > 0$, we can find an internal set $A \subseteq X$ such that $\delta(A) > \mu_\Omega(X)/2 > 0$.

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Theorem (2.13)

Let $(\Omega; \Sigma, \mu_\Omega)$ *be a Loeb space on a hyperfinite set* Ω *and* $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ *be a measurable function, i.e.,* $f^{-1}(O) \in \Sigma$ *for any open set* O *in* $\mathbb{R} \cup \{\pm\infty\}$*, then, there is an internal function* $F : \Omega \to {}^*\mathbb{R}$ *such that for almost all* $\omega \in \Omega$ *we have*

$$st(F(\omega)) = f(\omega).$$

Proof. Let $\mathcal{U} := \{O_n \mid n \in \mathbb{N}\}$ be a topological basis of $\mathbb{R} \cup \{\pm\infty\}$. For each $O_n \in \mathcal{U}$ let $A_{n,m} \subseteq f^{-1}(O_n)$ be increasing with respect to m such that $\lim_{m\to\infty} \mu_\Omega(A_m) = \mu_\Omega(f^{-1}(O_n))$.

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For each $m \in \mathbb{N}$ let

$$\mathcal{G}_m := \left\{ g : \bigcup_{n<m} A_{n,m} \to {}^*\mathbb{R} \mid g \text{ is internal and } g[A_{n,m}] \subseteq {}^*O_n \right\}.$$

It is easy to see that $\mathcal{G}_m$ is nonempty, internal, and decreasing. By countable saturation there is an $F \in \bigcap_{m\in\mathbb{N}} \mathcal{G}_m$. Note that the set

$$Z := \bigcup_{n\in\mathbb{N}} \left(f^{-1}(O_n) \setminus \bigcup_{m\in\mathbb{N}} A_{n,m} \right)$$

is a countable union of Loeb measure zero sets. Hence, $\mu_\Omega(Z) = 0$. For each $\omega \in \Omega \setminus Z$ and $O_n \in \mathcal{U}$, if $f(\omega) \in O_n$, then $\omega \in A_{n,m}$ for some $m > n$. Hence, $F(\omega) \in {}^*O_n$ which implies $st(F(\omega)) = f(\omega)$. □

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We present an application of nonstandard analysis to finance theory due to Dr. Yeneng Sun. This application may technically be the simplest one among all Dr. Sun's contributions to mathematical economics.

Given two hyperfinite Loeb spaces $(\Omega; \Sigma, \mu_\Omega)$ and $(\Psi; \Gamma, \nu_\Psi)$, one can form two different product measure spaces on $\Omega \times \Psi$.

The first one is the standard product measure space. For any two standard probability spaces $(\Omega; \Sigma, \mu)$ and $(\Psi; \Gamma, \nu)$ a rectangle is a set of form $A \times B$ for some $A \in \Sigma$ and $B \in \Gamma$. The measure $\mu \times \nu(A \times B) := \mu(A) \cdot \nu(B)$. Let $\Sigma \times \Gamma$ be the collection of all finite union of disjoint rectangles. The measure $\mu \times \nu$ can be trivially generalized to sets in $\Sigma \times \Gamma$. Note that

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Let's consider the product space of two hyperfinite Loeb spaces $(\Omega; \Sigma, \mu_\Omega)$ and $(\Psi; \Gamma, \nu_\Psi)$. Since $\Omega \times \Psi$ is again a hyperfinite set, one can form the Loeb probability space generalized by the normalized counting measure on all internal subsets of $\Omega \times \Psi$.

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Denote this Loeb product space by

$$(\Omega \times \Psi; \Sigma \otimes \Gamma, \mu_\Omega \otimes \nu_\Psi).$$

Since a finite union of disjoint rectangles is an internal subset of $\Omega \times \Psi$, we have that $\Sigma \times \Gamma \subseteq \Sigma \otimes \Gamma$. Since $\Sigma \otimes \Gamma$ is a σ-algebra and contains all subsets of zero-measure sets with respect to $\mu_\Omega \otimes \nu_\Psi$, we have that

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Theorem (2.14, Keisler's Fubini Theorem)

Let $(\Omega;\Sigma,\mu_\Omega)$ *and* $(\Psi;\Gamma,\nu_\Psi)$ *be two Loeb spaces. Assume that* $f:\Omega\times\Psi\to\mathbb{R}$ *is an integrable function on the Loeb product space* $(\Omega\times\Gamma,\Sigma\otimes\Gamma,\mu_\Omega\otimes\nu_\Psi)$*. Then,*

1. *for* ν_Ψ*-almost all* $y\in\Psi$, $f_y(x):=f(x,y)$ *is* μ_Ω*-integrable,*
2. $F(y):=\int_\Omega f(x,y)d\mu_\Omega(x)$ *is* ν_Ψ*-integrable, and*
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Imagine that an insurance company has a life insurance policy for people satisfying certain conditions. Each policy could bring a gain or loss of some values for the company with certain probability distribution. It is a common sense that if the identical policy is sold to enough many policy holders and each of these policy holders lives an independent life, then the company's financial risk of selling the policy can be diminished.

How can this phenomenon be mathematically modeled?

Definition (2.15)

Fix a probability space $(\Omega; \Sigma, \mu)$. A random variable is a measurable function $v(\omega) : \Omega \to \mathbb{R}$.

1. *By an individual insurance agent (for example, an insurance policy holder) we mean a random variable $f_i(\omega) : \Omega \to \mathbb{R}$.*
2. *By an insurance system we mean a function $f : \Omega \times I \to \mathbb{R}$ such that $f_i(\omega) := f(\omega, i)$ for each $i \in I$ is an insurance agent.*

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To find an idealize the model of the phenomenon, the number of insurance agents $|I|$ should be infinite. To measure the size of agent groups, a measure on the set I is needed. Since a measure should be countably additive, the size of I should be uncountable.

Definition (2.16)

Let $(\Omega; \Sigma, \mu)$ and $(\Psi; \Gamma, \nu)$ be two probability spaces.

1. *A function $f : \Omega \times \Psi \to \mathbb{R}$ is said to be jointly measurable if f is measurable with respect to the standard product space $(\Omega \times \Psi; \sigma(\Sigma \times \Gamma), \mu \times \nu)$;*
2. *Suppose a function $f : \Omega \times \Psi \to \mathbb{R}$ satisfies that $f^{\omega}(i) := f(\omega, i)$ is $(\Psi; \Gamma, \nu)$ measurable for almost every $\omega \in \Omega$ and $f_i(\omega) := f(\omega, i)$ is $(\Omega; \Sigma, \mu)$ measurable for almost every $i \in \Psi$. The function f is almost pairwise independent on Ψ if for $\nu \times \nu$-almost all pairs $(i, i') \in \Psi \times \Psi$, the random variables $f_i(\omega)$ and $f_{i'}(\omega)$ are independent.*

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2. *Suppose a function $f : \Omega \times \Psi \to \mathbb{R}$ satisfies that $f^{\omega}(i) := f(\omega, i)$ is $(\Psi; \Gamma, \nu)$ measurable for almost every $\omega \in \Omega$ and $f_i(\omega) := f(\omega, i)$ is $(\Omega; \Sigma, \mu)$ measurable for almost every $i \in \Psi$. The function f is almost pairwise independent on Ψ if for $\nu \times \nu$-almost all pairs $(i, i') \in \Psi \times \Psi$, the random variables $f_i(\omega)$ and $f_{i'}(\omega)$ are independent.*

Theorem (2.17, Joseph L. Doob)

Let $(\Omega; \Sigma, \mu)$ *and* $(\Psi; \Gamma, \nu)$ *be two probability spaces and* $f : \Omega \times \Psi \to \mathbb{R}$ *be a function such that*

1. f *is jointly measurable and square-integrable;*
2. f *is almost pairwise independent on* Ψ.

Then, for ν*-almost all* $i \in \Psi$, *the random variable* $f_i(\omega)$ *is* μ*-almost surely a constant function.*

By Theorem 2.17 there is no non-trivial insurance system can be jointly measurable with respect to the standard product of the insurance policy space and the space of insurance agents which are pairwise independent.

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Example (2.18)

Let N be a hyperfinite integer and $\Omega = \{\omega \mid \omega : [N] \to [2]\}$. Then, Ω is a hyperfinite set and $|\Omega| = 2^N$. Let $(\Omega; \Sigma, \mu_\Omega)$ be the Loeb space on Ω. Let $\Psi = [N]$ and $(\Psi; \Gamma, \nu_\Psi)$ be the Loeb space on Ψ. For each $i \in \Psi$ let $f_i : \Omega \to \mathbb{R}$ be defined as $f_i(\omega) := \omega(i)$. Then each f_i is a $0, 1$-valued random variable on Ω and

$$\mu_\Omega(\{\omega \mid f_i(\omega) = 0\}) = 1/2.$$

Each f_i can be viewed as a coin flip.

For any $i \neq i'$ in T, f_i and $f_{i'}$ are independent and have identical probability distribution.

Clearly, $f(\omega, i) := f_i(\omega)$ defines a measurable function on the Loeb product $(\Omega \times \Psi; \Sigma \otimes \Gamma, \mu_\Omega \otimes \nu_\Psi)$ such that all f_i are non-trivial.

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Theorem (2.19, Y. Sun)

Let $(\Omega;\Sigma,\mu_\Omega)$ and $(\Psi;\Gamma,\nu_\Psi)$ be two Loeb spaces and $f:\Omega\times\Psi\to\mathbb{R}$ be a square-integrable measurable insurance system in $(\Omega\times\Psi,\Sigma\otimes\Gamma,\mu_\Omega\otimes\nu_\Psi)$. If the insurance agents f_i and $f_{i'}$ are independent for almost all (i,i') in $\Psi\times\Psi$, then for almost all $\omega\in\Omega$

$$\int_\Psi f(\omega,i)d\nu_\Psi=\int_{\Psi\times\Omega} f(\omega,i)d\mu_\Omega\otimes\nu_\Psi=\int_\Psi\int_\Omega f(\omega,i)d\mu_\Omega d\nu_\Psi.$$

The theorem above is called the Exact Law of Large Numbers which indicates that the average pay-off of all insurance agents under particular realization ω for almost all $\omega\in\Omega$ is a constant which is the average pay-off of one agent.

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The End of Day Two

Thank you for your attention.