Exercise Solutions, 2023 Fudan Logic Summer School

Nonstandard Analysis and Combinatorial Number Theory

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Exercise Solutions

1.4.1 Prove that $1/[Id]$ is a non-zero infinitesimal in ${}^*\mathcal{R}$ as defined in Example 1.11.

Proof: Given any $m \in \mathbb{N}$, it suffices to show that $1/[Id] < 1/m$, which is equivalent to $[Id] > m$. Since $\{n \in \mathbb{N} \mid Id(n) > m\} = \{m+1, m+2, \ldots\} \in \mathcal{F}$, we have $[Id] > m$. □

1.4.2 Prove that a standard sequence s of real numbers being convergent as defined in Definition 1.1.5 is equivalent to that s is a Cauchy sequence in the standard sense.

Proof: "⇒": Suppose that the sequence s converges in the sense of Definition 1.15. Given any $\epsilon > 0$, let φ be the sentence that there exists an $m \in {}^*\mathbb{N}$ be the least such that for all $n, n' > m$ we have $|{}^*s(n) - {}^*s(n')| < \epsilon$. Clearly, φ is true in ${}^*\mathcal{R}$ because any hyperfinite integer witnesses the existence of m. Hence, φ is true in $\mathcal{R}$ by the transfer principle. So, s is a Cauchy sequence.

"⇐": Assume that s is a Cauchy sequence in the standard sense. Given any hyperfinite integers N, N', we show that ${}^*s(N) \approx {}^*s(N')$. So, for any standard $\epsilon > 0$, it suffices to show that $|{}^*s(N) - {}^*s(N')| < \epsilon$. Since s is Cauchy, there exists an $n_\epsilon \in \mathbb{N}$ such that sentence φ: $|s(n) - s(n')| < \epsilon$ for any $n, n' > n_\epsilon$ is true in $\mathcal{R}$. So the sentence φ is also true in ${}^*\mathcal{R}$. Since N, N' are hyperfinite, they are clearly greater than n_ϵ. Hence, $|{}^*s(N) - {}^*s(N')| < \epsilon$. □.

1.4.3 Let $f : [a, b] \to \mathbb{R}$ be a standard function. Prove that f being continuous at some $c \in (a, b)$ or uniformly continuous on $[a, b]$ in terms of Definition 1.17 is equivalent to that f is continuous at c or uniformly continuous on $[a, b]$, respectively, in the standard sense ($\epsilon - \delta$ definition).

Proof: Continuity: "⇒": Assume that f is continuous at c in the sense of Definition 1.17. Given standard $\epsilon > 0$, let φ be the sentence

$$\exists \delta > 0\, \forall x\, (|x - c| < \delta \to |{}^*f(x) - f(c)| < \epsilon).$$

Then, φ is true in ${}^*\mathcal{R}$ because any positive infinitesimal can be the witness of δ. Hence, φ is true in $\mathcal{R}$, i.e., there is a standard $\delta > 0$ such that $|x - c| < \delta \to |f(x) - f(c)| < \epsilon$ for any $x \in \mathbb{R}$.

"⇐": Assume that f is continuous at c in the standard sense. Given any $r \in {}^*\mathbb{R}$ with $r \approx c$. We show that ${}^*f(r) \approx f(c)$. Given an arbitrary standard real $\epsilon > 0$, it suffices to show that $|{}^*f(r) - f(c)| < \epsilon$. Since there is a standard $\delta > 0$ such that $|x - c| < \delta \to |{}^*f(x) - f(c)| < \epsilon$ for all $x \in {}^*\mathbb{R}$, we have $|{}^*f(r) - f(c)| < \epsilon$ because $r \approx c$ implies $|r - c| < \delta$. Since ϵ is arbitrary, we conclude that $|{}^*f(r) - f(c)| \approx 0$.

Uniform continuity: "⇒": Assume that f is uniformly continuous in $[a, b]$ in the standard sense. Given any $r, r' \in {}^*[a, b]$ with $r \approx r'$, we show that ${}^*f(r) \approx {}^*f(r')$. Given arbitrary standard $\epsilon > 0$, it suffices to show that $|{}^*f(r) - {}^*f(r')| < \epsilon$. Since there exists a standard $\delta > 0$ such that the sentence φ:

$$\forall x, y \in [a, b]\, (|x - y| < \delta \to |f(x) - f(y)| < \epsilon)$$

is true in $\mathcal{R}$, we have that φ is true in ${}^*\mathcal{R}$. Clearly, we have $|r - r'| \approx 0 < \delta$. Therefore, $|{}^*f(r) - {}^*f(r')| < \epsilon$.

"⇐": Assume that f is uniformly continuous in $[a, b]$ in the sense of Definition 1.17. Given standard $\epsilon > 0$, let φ be the sentence

$$\exists \delta > 0\, \forall x, y \in {}^*[a, b]\, (|x - y| < \delta \to |{}^*f(x) - {}^*f(y)| < \epsilon)$$

is true in ${}^*\mathcal{R}$ because any positive infinitesimal can be that δ. By the transfer principle we have that φ is true in $\mathcal{R}$, i.e.,

$$\exists \delta > 0\, \forall x, y \in [a, b]\, (|x - y| < \delta \to |f(x) - f(y)| < \epsilon)$$

is true in $\mathcal{R}$. □

1.4.4 Let $f : [a, b] \to \mathbb{R}$ be a standard bounded function. Prove that f being Riemann integrable on $[a, b]$ as defined in Definition 1.22 is equivalent to that f is Riemann integrable on $[a, b]$ in the standard sense.

Proof: We introduce some notation first. Given any partition P of $[a, b]$ let $U({}^*f, P)$ be the upper sum and $L({}^*f, P)$ be the lower sum as defined in some standard textbook. If T is a set of tag points of P let $S({}^*f, P, T)$ be the Riemann sum as defined in some standard textbook. For each positive $n \in {}^*\mathbb{N}$ let P_n be the partition $\{a, b\} \cup (\{z/n \mid z \in {}^*\mathbb{Z}\} \cap {}^*[a, b])$. If $P = \{a = x_0 < x_1 < \cdots < x_m = b\}$ is a partition of $[a, b]$ let $\|P\| := \max\{|x_i - x_{i-1}| \mid i = 1, 2, \ldots, m\}$.

"$\Rightarrow$": Assume that f is Riemann integrable on $[a, b]$ in the sense of Definition 1.22. By the assumption we have that $U({}^*f, P_K) - L({}^*f, P_K) \approx 0$. Given any standard $\epsilon > 0$ the sentence φ:

$$\exists n \in {}^*\mathbb{N}\, (U({}^*f, P_K) - L({}^*f, P_K) < \epsilon)$$

is true in ${}^*\mathcal{R}$ because K is a witness of n. By the transfer principle we have that φ is true in $\mathcal{R}$. This shows that f is Riemann integrable on $[a, b]$ in the standard sense.

"$\Leftarrow$": Assume that f is Riemann integrable in the standard sense. Then, there is a standard real I such that for any standard ϵ there exists a standard $\delta > 0$ such that the sentence φ:

$$\forall \text{ partition } P \text{ and set of tag points } T\, (\|P\| < \delta \to |S(f, P, T) - I| < \epsilon)$$

is true in $\mathcal{R}$. Hence, it is true in ${}^*\mathcal{R}$. Given any internal sets of tag points T, T' for P_K, since $\|P_K\| = 1/K < \delta$ we have

$$|S({}^*f, P_K, T) - S({}^*f, P_K, T')| \le |S({}^*f, P_K, T) - I| + |I - S({}^*f, P_K, T')| < 2\epsilon.$$

Since ϵ is arbitrary, we conclude that $S({}^*f, P_K, T) \approx S({}^*f, P_K, T')$. □

2.4.1 Let A be a set in $\mathcal{V}$. Prove that ${}^*A = \{{}^*a \mid a \in A\}$ iff A is a finite set.

Proof: "$\Rightarrow$": Suppose ${}^*A = \{{}^*a \mid a \in A\}$. If A is infinite, we can find a sequence of distinct elements $\{a_n \mid n \in \mathbb{N}\}$ in A. Let $S_n := {}^*A \setminus \{{}^*a_i \mid i = 1, 2, \ldots, n\}$. Then, S_n is nonempty and decreasing. By countable saturation there is an element b in every S_n for $n \in \mathbb{N}$. This b is in ${}^*A$ and different from any of these ${}^*a_n$'s. So, ${}^*A \neq \{{}^*a \mid a \in A\}$.

"$\Leftarrow$": Assume that $A := \{a_1, a_2, \ldots, a_n\}$ is a finite set. Then, the sentence φ:

$$\forall x \in A\, (x = a_1 \vee x = a_2 \vee \cdots \vee x = a_n)$$

is true in $\mathcal{V}$. By the transfer principle φ is true in ${}^*\mathcal{V}$, i.e., for any $b \in {}^*A$ the element b must be one of these ${}^*a_i$'s. □

2.4.2 Prove that an internal set $A \in {}^*\mathcal{V}$ is either finite or uncountable.

Proof: Suppose that the internal set A is countable and $\{a_n \mid n \in \mathbb{N}\}$ is an enumeration of A. Let $S_n := A \setminus \{a_i \mid i = 1, 2, \ldots, n\}$. Then, S_n is internal, nonempty, and decreasing. By countable saturation one can find an element b in every S_n for $n \in \mathbb{N}$. This contradicts that A is enumerated by these a_n's. □

2.4.3 Let N be a hyperfinite integer, $\Omega := \{j/N \mid j = 0, 1, \ldots, N-1\}$, and $(\Omega; \Sigma, \mu_\Omega)$ be the Loeb space on Ω. Note that $st \upharpoonright \Omega$ is a function from Ω to the standard unit interval $[0, 1]$ (cf. Definition 1.14). Let $\Gamma := \{U \subseteq [0, 1] \mid st^{-1}[U] \cap \Omega \in \Sigma\}$ and $\lambda(U) := \mu_\Omega(st^{-1}[U] \cap \Omega)$ for each $U \in \Gamma$. Prove that $([0, 1]; \Gamma, \lambda)$ is the Lebesgue measure space on $[0, 1]$.

Proof: We show first that every closed subinterval of $[0, 1]$ is in Γ. Let $[a, b] \subseteq [0, 1]$ and $X := st^{-1}([a, b])$. For each $n \in \mathbb{N}$ let $x_n, y_n \in \Omega$ such that $st(x_n) = a - 1/n$ and $st(y_n) = b + 1/n$. Then, $\mu_\Omega([x_n, y_n] \cap \Omega) = b - a + 2\epsilon$. Since $X = \bigcap\{[x_n, y_n] \cap \Omega \mid n \in \mathbb{N}\}$ we have that $X \in \Sigma$ and

$$\mu_\Omega(X) = \lim_{n \to \infty} \mu_\Omega([x_n, y_n] \cap \Omega) = b - a.$$

So, $[a, b] \in \Gamma$ and $\lambda([a, b]) = b - a$. It is easy to check that Γ is a σ-algebra and complete, we conclude that Γ contains all Lebesgue measurable subsets of $[0, 1]$. Next we show that every set $S \in \Gamma$ is Lebesgue measurable.

Notice a fact that if $A \subseteq \Omega$ is internal, then $st[A]$ is a closed subset of $[0, 1]$. Indeed, if s is a standard convergent sequence in S with limit α and $st(a_n) = s(n)$ for some $a_n \in A$, then the sequence $\{a_n\}$ can be extended to a hyperfinite sequence $\{a_n \in A \mid 0 \leq n < N\}$ by countable saturation. For any hyperfinite integer $K < N$ it is easy to check that $st(a_K) = \alpha$. Hence, $\alpha \in S$.

Given $S \in \Gamma$ with $\lambda(S) = \alpha > 0$, let $X = st^{-1}(S) \in \Sigma$. Note that $\mu_\Omega(X) = \alpha$. Let $B_n \subseteq X$ be internal such that $\mu_\Omega(B_n) > \alpha - 1/n$. Then,

$$\alpha = \mu_\Omega(X) \geq \mu_\Omega(st^{-1}(st(B_n))) = \lambda(st(B_n)) \geq \mu_\Omega(B_n) > \alpha - \frac{1}{n}.$$

So, S contains a countable union U of closed sets $st(B_n)$ in $[0,1]$ such that

$$\lambda(S) = \lambda\left(\bigcup_{n=1}^{\infty} st(B_n)\right).$$

It suffices now to show that every λ-measure zero set is Lebesgue measurable.

Suppose $\alpha = 0$. We show that S is a subset of a Borel set with λ measure zero. We consider the complement S^c of S in $[0,1]$. Since $\lambda(S^c) = 1$, we can find a sequence of closed sets $C_n = st(B_n) \subseteq S^c$ as done above such that

$$\lambda\left(\bigcup_{n=1}^{\infty} C_n\right) = 1.$$

Hence, S is a subset of the complement of the union of these C_n's which is Borel set and has λ-measure zero. □

2.4.4 Let $(\Omega; \Sigma, \mu_\Omega)$ and $(\Psi; \Gamma, \nu_\Psi)$ be two Loeb spaces defined in Example 2.18. Let

$$A := \{(\omega, i) \in \Omega \times \Psi \mid \omega(i) = 0\}.$$

Note that $A \in \Sigma \otimes \Gamma$ because A is internal. Prove that $\mu_\Omega \otimes \nu_\Psi\,(A) = 1/2$ and $A \notin \sigma(\Sigma \times \Gamma)$.

Proof: For each $\omega \in \Omega$ let $\omega' : [N] \to [2]$ be such that $\omega'(i) = 1 - \omega(i)$ for every $i \in [N]$. Then, $\{(\omega', i) \mid (\omega, i) \in A\} = A^c$. So, A contains exactly half of the elements in $\Omega \times \Psi$. So, $\mu_\Omega \otimes \mu_\Psi(A) = 1/2$.

Suppose $A \in \sigma(\Sigma \times \Gamma)$. Then we can find at least one set $B_1 \times B_2 \in \Sigma \times \Gamma$ such that $B_1 \times B_2 \subseteq A$, $\mu_\Omega(B_1) > 0$, and $\mu_\Psi(B_2) > 0$. Note that if $(\omega, i) \in B_1 \times B_2$, then $\omega(i) = 0$. Note also that if $i_1 < i_2 < \cdots < i_m$ are in Ψ, there are exactly $2^N/2^n = 2^{N-n}$ many $\omega \in \Omega$ with $\omega(i_j) = 0$ for $j = 1, 2, \ldots, m$. If B_2 has positive measure, $|B_2|$ must be hyperfinite. Hence, B_1 can contain at most $2^{N-|B_2|}$ elements. This show that $\mu_\Omega(B_1) = 0$, which is a contradiction. □

3.4.1 Let $A \subseteq \mathbb{N}$. Prove that the limit of the sequence

$$s_n := \sup_{k \in \mathbb{N}} \frac{|A \cap (k + [n])|}{n}$$

as $n \to \infty$ exists.

Proof: Suppose the limit does not exist. We can find two subsequences s_{n_i} with limit α and s_{m_i} with limit β. Assume that $d = \beta - \alpha > 0$. Choose an $s_{n_{i_0}} < \alpha + d/4$ and choose sufficiently large m_{i_0} such that $s_{m_{i_0}} > \beta - d/4$ and $n_{i_0}/m_{i_0} < d/4$. Partition $[m_{i_0}]$ into subintervals of length n_{i_0} except the last one which has length between 0 and $n_{i_0} - 1$. Let $\mathcal{I}$ be the collections of these subintervals of length n_{i_0} and J be the last one with length $< n_{i_0}$. By the maximality of $s_{n_{i_0}}$ we have that

$$\frac{|I \cap A|}{n_{i_0}} \leq s_{n_{i_0}}$$

for every $I \in \mathcal{I}$. Therefore, we have that

$$s_{m_{i_0}} \leq s_{n_{i_0}} + \frac{|J \cap A|}{m_{i_0}} < \alpha + \frac{d}{2} < \beta - \frac{d}{4}$$

which contrdicts the choice of m_{i_0}. □

3.4.2 Prove Lemma 3.4: Let $A \subseteq \mathbb{N}$. Then, $\overline{BD}(A)$ is the largest real α in $[0, 1]$ such that there exist $k_m, n_m \in \mathbb{N}$ with $n_m \to \infty$ as $m \to \infty$ such that

$$\lim_{m\to\infty} \frac{|A \cap (k_m + [n_m])|}{n_m} = \alpha.$$

Proof: Let $\overline{BD}(A) = \alpha$. For each $m \in \mathbb{N}$ choose an interval $k_m + [n_m]$ such that $n_m > m$ and

$$\frac{|A \cap (k_m + [n_m])|}{n_m} \geq \alpha - \frac{1}{m}.$$

Note that n_m exists by definitions. Clearly,

$$\alpha = \lim_{n\to\infty} s_n = \alpha \geq \lim_{m\to\infty} \frac{|A \cap (k_m + [n_m])|}{n_m} = \alpha.$$

If there is a sequence of interval $k_m + [n_m]$ with $n_m \to \infty$ such that

$$\lim_{m\to\infty} \frac{|A \cap (k_m + [n_m])|}{n_m} = \beta,$$

then $\beta \leq \alpha$ because

$$\frac{|A \cap (k_m + [n_m])|}{n_m} \leq \sup_{k\in\mathbb{N}} \frac{|A \cap (k + [n_m])|}{n_m} \to \alpha.$$

□

3.4.3 Prove Part 2 of Proposition 3.5: Let $A \subseteq \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then $\overline{d}(A) \geq \alpha$ iff $\frac{|{}^*A \cap [N]|}{N} \gtrapprox \alpha$ for some hyperfinite integer N.

Proof. "$\Rightarrow$": Given standard $m \in \mathbb{N}$, there is an $n_m \in \mathbb{N}$ such that $n_m > m$ and $\frac{|A \cap [n_m]|}{n_m} > \alpha - 1/m$. By the overspill principle, we can find a hyperfinite integer K such that $n_K > K$ and

$$\frac{|A \cap [n_K]|}{n_K} > \alpha - \frac{1}{K}.$$

Now $N = n_K$ works.

"$\Leftarrow$": Given any standard $\epsilon > 0$ and $m \in \mathbb{N}$, the sentence φ:

$$\exists N \in {}^*\mathbb{N} \left(N > m \wedge \frac{|{}^*A \cap [N]|}{N} > \alpha - \epsilon \right)$$

is true in ${}^*\mathcal{V}$. By the transfer principle, φ is true in $\mathcal{V}$, i.e.,

$$\exists n \in \mathbb{N} \left(n > m \wedge \frac{|A \cap [n]|}{n} > \alpha - \epsilon \right).$$

This proves that $\overline{d}(A) \geq \alpha$. □

3.4.4 Prove that Theorem 3.14 using Theorem 3.13 and By-one-get-one-free Thesis, i.e., prove that if B is a piecewise asymptotic basis of piecewise asymptotic average order h^*_{pa}, then for any $A \subseteq \mathbb{N}$ we have

$$\overline{BD}(A+B) \geq \overline{BD}(A) + \frac{1}{2h^*_{pa}} \overline{BD}(A)(1 - \overline{BD}(A)).$$

Proof. Let B be a piecewise asymptotic basis of piecewise asymptotic average order h^*_{pa}. Given $\epsilon > 0$, there exists a suitable sequence $\mathcal{I}$ of intervals such that

$$h^*_{pa} + \frac{\epsilon}{2} \geqslant h^*_{\mathcal{I}}.$$

By the overspill principle, there exists an interval $k + [K]$ of hyperfinite length and $n_0 \in \mathbb{N}$ such that

$$\sup_{n_0+k \leqslant l \leqslant K+k} \frac{1}{l - n_0 - k + 1} \sum_{i=n_0+k}^{l} h_{k+[K]}(i) \leqslant h^*_{\mathcal{I}} + \frac{\epsilon}{2}$$

where $h_{k+[K]}(i) = \min\{h' \in {}^*\mathbb{N} : i \in h'(({}^*\mathcal{B} - k) \cap {}^*\mathbb{N}) + k)\}$. Obviously, the asymptotic average order $\bar{h}$ of the asymptotic basis $({}^*\mathcal{B} - H) \cap \mathbb{N}$ satisfies

$$\bar{h} \leqslant h^*_{\mathcal{I}} + \frac{\epsilon}{2} \leqslant h^*_{pa} + \epsilon.$$

Let $\overline{BD}(A) = \alpha$. Then, there exists an $a \in {}^*\mathbb{N}$ such that $\underline{d}({}^*\!A - a) = \alpha$. Applying Rohrbach's theorem, we have

$$\underline{d}(({}^*\!A - a) + ({}^*\mathcal{B} - k)) = \underline{d}(({}^*\!A + {}^*\mathcal{B}) - (a + k)) \geqslant \alpha + \frac{1}{2(h^*_{pa} + \epsilon)}\alpha(1 - \alpha)$$

which implies that

$$\overline{BD}(A + B) \geqslant \overline{BD}(A) + \frac{1}{2(h^*_{pa} + \epsilon)}\overline{BD}(A)(1 - \overline{BD}(A)).$$

Since ϵ can be arbitrarily small, the conclusion follows. □

4.4.1 Prove that the map $i_{1,1} : \mathcal{V}_1 \to \mathcal{V}_2$ defined in Definition 4.2 is an elementary embedding.

Proof: It suffices to prove the correspondent version of Łoś Theorem, i.e., for any formula $\varphi(\overline{x})$ and any $\overline{f} \in \mathcal{V}_1^{\mathbb{N}_1} \cap \mathcal{V}_1$ we have $\mathcal{V}_2 \models \varphi(\overline{[f]}_{\mathcal{F}_1})$ iff $\{n \in \mathbb{N}_1 \mid \mathcal{V}_1 \models \varphi(\overline{f(n)})\} \in \mathcal{F}_1$. The steps of the proof is the same as the steps for proving usual Łoś Theorem except one case when a formula $\varphi(\overline{x})$ has the form $\exists y\, \psi(y, \overline{x})$. Note that $i_{1,1}(a) = [\phi_a]_{\mathcal{F}_1}$.

Case 1: $\varphi(\overline{x})$ is an atomic formula $x \in y$. Let $[f]_{\mathcal{F}_1}, [g]_{\mathcal{F}_1} \in \mathcal{V}_2$. It is clear by definition that $[f]_{\mathcal{F}_1} {}^*\!\!\in [g]_{\mathcal{F}_1}$ iff $\{n \in \mathbb{N}_1 \mid f(n) \in g(n)\} \in \mathcal{F}_1$ by definition.

Case 2: $\varphi(\overline{x})$ is $\psi(\overline{x}) \wedge \chi(\overline{x})$. Let $\overline{[f]_{\mathcal{F}_1}} \in \mathcal{V}_2$. Then, $\mathcal{V}_2 \models \psi(\overline{[f]}_{\mathcal{F}_1})$ iff $\{n \in \mathbb{N}_1 \mid \mathcal{V}_1 \models \psi(\overline{f(n)})\} \in \mathcal{F}_1$ and $\mathcal{V}_2 \models \chi(\overline{[f]}_{\mathcal{F}_1})$ iff $\{n \in \mathbb{N}_1 \mid \mathcal{V}_1 \models \chi(\overline{f(n)})\} \in \mathcal{F}_1$ by the induction hypothesis.

Case 3: $\varphi(\overline{x})$ is $\exists y\, \psi(y, \overline{x})$. Let $\overline{[f]_{\mathcal{F}_1}} \in \mathcal{V}_2$. If $\mathcal{V}_2 \models \varphi(\overline{[f]}_{\mathcal{F}_1})$, then there is a $[g]_{\mathcal{F}_1} \in \mathcal{V}_2$ such that $\mathcal{V}_2 \models \psi([g]_{\mathcal{F}_1}, \overline{[f]}_{\mathcal{F}_1})$. Because ψ has lower complexity than φ has, we have, by the induction hypothesis, that $\{n \in \mathbb{N}_1 \mid \mathcal{V}_1 \models \psi(g(n), \overline{f(n)})\} \in \mathcal{F}_1$. Since $\mathcal{V}_1 \models \psi(g(n), \overline{f(n)})$ implies $\mathcal{V}_1 \models \exists y\, \psi(y, \overline{f(n)})$ we have that $\{n \in \mathbb{N}_1 \mid \mathcal{V}_1 \models \varphi(\overline{f(n)})\} = \{n \in \mathbb{N}_1 \mid \mathcal{V}_1 \models \exists y\, \psi(y, \overline{f(n)})\} \in \mathcal{F}_1$.

On the other hand, we can assume that $F := \{n \in \mathbb{N}_1 \mid \mathcal{V}_1 \models \varphi(\overline{[f]}_{\mathcal{F}_1})\} \in \mathcal{F}_1$. By the axiom of choice the set $\mathcal{V}_0$ has a well order $\lhd$. By taking the ultrapower of the well order $\lhd$ we have that $\mathcal{V}_1$ has an internal well order ${}^*\!\lhd$.

For each $n \in F$ let $g(n)$ be the ${}^*\lhd$-least a such that $\mathcal{V}_1 \models \psi(a, \overline{f(n)})$ and $g(n) = 0$ for any $n \in \mathbb{N} \setminus F$. Then, $g \in \mathcal{V}_1$ because it can be defined by a first order formula with internal parameters. Clearly, $\{n \in \mathbb{N}_1 \mid \mathcal{V}_1 \models \psi(g(n), \overline{f(n)})\} \in \mathcal{F}_1$ because it is a$\mathcal{V}_1$-internal set and contains F. By induction hypothesis we have $\mathcal{V}_2 \models \psi([g]_{\mathcal{F}_1}, \overline{[f]_{\mathcal{F}_1}})$ which implies $\mathcal{V}_2 \models \exists y\, \psi(y, \overline{[f]_{\mathcal{F}_1}})$, i.e., $\mathcal{V}_2 \models \varphi(\overline{[f]_{\mathcal{F}_1}})$.

For a formula $\varphi(\overline{x})$ and $\overline{a} \in \mathcal{V}_0$ we have that $\mathcal{V}_0 \models \varphi(\overline{a})$ iff

$$\{n \in \mathbb{N}_1 \mid \mathcal{V}_0 \models \varphi(\overline{\phi_a(n)})\} = \mathbb{N}_1 \in \mathcal{F}_1$$

iff $\mathcal{V}_2 \models \varphi(\overline{[\phi_a]})$ by the above version of Łoś' Theorem. □

4.4.2 Let $s \in \mathbb{N}_0$ and $\Gamma(\overline{x})$ be a countable collection of formulas with parameters from $\mathcal{V}_0$ and s free variables $\overline{x} = (x_1, x_2, \ldots, x_s)$. Prove that there exists a homothetic copy $HC_{\vec{a},d}$ of $[N]^s$ for some $N \in \mathbb{N}_1 \setminus \mathbb{N}_0$ such that for any $0 \le l, l' < N^s$

$$\mathcal{V}_1 \models \varphi(HC_{\vec{a},d}(l)) \iff \mathcal{V}_1 \models \varphi(HC_{\vec{a},d}(l'))$$

for every formula φ in Γ.

Proof: Let $\{\varphi_i(\overline{x}) \mid i \in \mathbb{N}_0\}$ be an enumeration of Γ. For each i let $B_i := \{\overline{b} \in [\mathbb{N}_0]^s \mid \mathcal{V}_0 \models \varphi_i(\overline{b})\}$. Then, B_i and B_i^c form a partition of $[\mathbb{N}_0]^s$. For each $n \in \mathbb{N}_0$ the set $[\mathbb{N}_0]^s$ can be partitioned into 2^n parts by taking all intersections of B_i or B_i^c for each i for $i = 0, 1, \ldots, n-1$. By the multidimensional van der Waerden's Theorem we can find a homothetic copy $HC_{\vec{a}_n,d_n,n}$ of $[n]^s$ entirely in one part of the partition. Note that $\mathcal{V}_0 \models \varphi_i(HC_{\vec{a}_n,d_n,n}(l))$ iff $\mathcal{V}_0 \models \varphi_i(HC_{\vec{a}_n,d_n,n}(l'))$ for any $i = 0, 1, \ldots, n-1$ and $0 \le l, l' < n^s$. By countable saturation we can find a hyperinteger N and a homothetic copy $HC_{\vec{a},d,N} \subseteq [\mathbb{N}_1]^s$ of $[N]^s$ such that $\mathcal{V}_1 \models \varphi_i(HC_{\vec{a},d,N}(l))$ iff $\mathcal{V}_1 \models \varphi_i(HC_{\vec{a},d,N}(l'))$ for any $\varphi_i(\overline{x}) \in \Gamma$ and $0 \le l, l' < N^s$. □

4.4.3 Let $r \in \mathbb{R}_2$ and $|r| < \alpha$ for some $\alpha \in \mathbb{R}_0$. Prove that $st(st_1(r)) = st(r)$.

Proof: Set $r' := st_1(r) \in \mathbb{R}_1$ and $r'' := st(r') \in \mathbb{R}_0$. It suffices to show that $st(r) = r''$. We want to show that $|r - r''| < 1/n$ for every $n \in \mathbb{N}_0$. Bote that $|r - r'| < 1/n$ for every $n \in \mathbb{N}_1$ and $|r' - r''| < 1/n$ for every $n \in \mathbb{N}_0$. Since $\mathbb{N}_0 \subseteq \mathbb{N}_1$, given any $n \in \mathbb{N}_0$ the number n is also in $\mathbb{N}_1$. Hence,

$$|r - r''| \le |r - r'| + |r' - r''| < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.$$

This shows that $st(r) = r''$. □

4.4.4 Prove that the set

$$C := \left\{\mu^1_{|P|}(A \cap P) \mid P \text{ is an } a.p. \text{ and } |P| \in \mathbb{N}_2 \setminus \mathbb{N}_1\right\}$$

is an element in $\mathcal{V}_1$ where $A \subseteq \mathbb{N}_2$ is $\mathcal{V}_2$-internal.

Proof: Let φ be the following sentence

$$\begin{aligned}&\forall A \in \mathscr{P}_1(\mathbb{N}_1)\, \exists C \in \mathscr{P}_0(\mathbb{R}_0)\, \forall x \in \mathbb{R}_0\, (x \in C \\ &\quad \longleftrightarrow \exists P\, (P \text{ is an a.p.} \wedge |P| \in \mathbb{N}_1 \setminus \mathbb{N}_0 \wedge x = \mu_{|P|}(A \cap P))\end{aligned}$$

where $\mathscr{P}_j$ means the power set operator in $\mathcal{V}_j$. Note that φ is true in $(\mathcal{V}_1; \mathbb{R}_0)$ because every subset of $\mathbb{R}_0$ is an element in $\mathcal{V}_0$. Since $i_{0,1}$ is an elementary embedding from $(\mathcal{V}_1; \mathbb{R}_0)$ to $(\mathcal{V}_2; \mathbb{R}_1)$ we have that the sentence

$$\begin{aligned}&\forall A \in \mathscr{P}_2(\mathbb{N}_2)\, \exists C \in \mathscr{P}_1(\mathbb{R}_1)\, \forall x \in \mathbb{R}_1\, (x \in C \\ &\quad \longleftrightarrow \exists P\, (P \text{ is an a.p.} \wedge |P| \in \mathbb{N}_2 \setminus \mathbb{N}_1 \wedge x = \mu^1_{|P|}(A \cap P))\end{aligned}$$

is true in $\mathcal{V}_2$. Therefore, the set

$$C := \left\{\mu^1_{|P|}(A \cap P) \mid P \text{ is an } a.p. \text{ and } |P| \in \mathbb{N}_2 \setminus \mathbb{N}_1\right\}$$

is in $\mathcal{V}_1$. □