From Stone Duality to Classifying Topos

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Methodology: Conceptual Thinking

Definition vs. Conception

A common pattern in maths textbooks:

1. Define some concept C,
2. Prove some theorems about (sometimes *characterize*) C.

But when maths is *being developed*, we often see a different pattern:

1. Start with some *intuition* about some to-be-defined concept C,
2. Try to capture C with some definitions,
3. Justify (or refute) the definition by proving theorems about C that corresponds to our initial intuition.

Definition vs. Conception

I argue: the textbook pattern can be *misleading*. Students might think:

- The textbook definition of C is the *correct conception*,
- The theorems about C, even those who *characterize* C, is just some properties that C happens to enjoy.

This is usually *not* the case. We see such examples in category theory everyday.

Example: Product

One simple example is Cartesian product of sets. Textbooks might say:

1. Given two sets X, Y, their Cartesian product $X \times Y$ consists of elements of the form $\{\{x\}, \{x, y\}\}$ with $x \in X, y \in Y$.
2. $X \times Y$ and canonical projections $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ make $X \times Y$ the *categorical* product of X, Y in the category Set.

Problem:

- For students, $X \times Y$ *is* the set of $\{\{x\}, \{x, y\}\}$, which happens to have that universal property.
- In reality, the universal property is the *essense* behind the idea of product, while $\{\{x\}, \{x, y\}\}$ is just one of many ways to make it work.

Definition vs. Conception

One should often *invert* the textbook process. We look for the *correct conception* first, then fit the original definition into our conception.

This is largely the style of William Lawvere who introduced *category theory* into the discussion of mathematical logic and foundation. Category theory often plays a *normative* role here: if one can describe something concisely using category theory (like how Lawvere uses adjoint functor), he's probably doing the *right* thing.

Summary

Two points:

- Correct conception > technical definition.
- Using category theory as a norm.

In this talk, we will heavily adopt this style:

1. I'll present the textbook definitions and theorems first,
2. Then formulate a conceptual picture with them,
3. Re-name several things to help our intuition,
4. Reach the "correct" (although debatable) conception which often differs from our original definition.

Classical Stone Duality

Stone's Insight

- A *Boolean algebra* is a bounded distributive lattice s.t. every element has a complement.
- A *Stone space* is a compact, Hausdorff, totally disconnected topological space.

We write Bool for the category of Boolean algebras and homomorphisms, Stone for the category of Stone spaces and continuous mappings.

Stone's Insight

In 1936 (probably even before), Marshall Stone discovered:

- Every Boolean algebra A induces a Stone space $\mathrm{St}(A)$ of its ultrafilters.
- Every Stone space X induces a Boolean algebra $\mathrm{Cl}(X)$ of its clopen sets.
- St and Cl are both *functors*:

$$\mathrm{St} : \mathsf{Bool}^{\mathrm{op}} \rightleftarrows \mathsf{Stone} : \mathrm{Cl}$$

- Moreover, these functors comprise an *equivalence* between Bool and Stone.

I find it interesting that Stone's discovery, something *impossible to state without category theory*, predates the birth of category theory.

Boolean Algebra and Logic

It's well-known that Boolean algebra is a model of classical propositional logic:

- Given a propositional theory T and a Boolean algebra A, it makes sense to talk about T's *model* in A. Let $\mathrm{Mod}_T(A)$ denote the set of T's model in A.
- A homomorphism $\varphi : A \to B$ induces a function $\mathrm{Mod}_T(A) \to \mathrm{Mod}_T(B)$, since all the relevant logical structures are preserved.
- Categorically speaking, Mod_T is a *functor* Bool $\to$ Set.

Boolean Algebra and Logic

- Every propositional theory T has a *Lindenbaum algebra* $\mathcal{S}[T] \in$ Bool, obtained by equalizing every provably equivalent formulas.
- Crucial point: $\mathcal{S}[T]$ *represents the functor* Mod_T.
- This means: there's a *generic* model of T in $\mathcal{S}[T]$[1], such that any model of T in any Boolean algebra A can be lifted to a homomorphism $\mathcal{S}[T] \to A$.

$$\text{Bool}(\mathcal{S}[T], -) \cong \mathrm{Mod}_T : \text{Bool} \to \text{Set}.$$

[1] Concretely, it interprets every formula into its equivalence class.

The Logical Nature of Boolean Algebra

- Moreover, every Boolean algebra A is the Lindenbaum algebra of *some* propositional theory T.[2]
- Conceptually, for any propositional theory T, we think of $\mathcal{S}[T]$ as T's *essential syntactic content*, where T itself is just a *presentation* or a user interface.
- A homomorphism $\varphi : [T_1] \to [T_2]$ should be seen as some *translation* of T_1 into T_2.
- Thus, Bool is *the category of propositional theories and translations*.

[2] You can try to figure out what T can be. There could be many answers.

Invert the Process

Now comes the cool step: we *define* Stone to be the opposite category of Bool.

$$\text{Stone} := \text{Bool}^{\text{op}}.$$

We call the canonical (in fact *identity*) functor $\text{Bool}^{\text{op}} \rightleftarrows \text{Stone}$ with the original name St, Cl, and we *think of* Stone as a category of spaces and continuous mappings.

- Every homomorphism $A \to B$ thus becomes (conceptually) a continuous mapping $\mathrm{St}(B) \to \mathrm{St}(A)$.
- Bool's initial $2 = \{\top, \bot\}$ and terminal 1 thus becomes Stone's terminal $\bullet = \mathrm{St}(2)$ and initial $\varnothing = \mathrm{St}(1)$.
- Like any rich category of spaces, we think of the initial $\varnothing$ as the *empty space* and the terminal $\bullet$ as the *singleton space*.

Space of *What?*

- For any space X, a *point* $x \in X$ is exactly a continuous mapping $x : \bullet \to X$.
- In our case, this corresponds to a homomorphism $x^* : \mathrm{Cl}(X) \to 2$, which is a *model* of $\mathrm{Cl}(X)$ in 2. Note that this is the classical notion of model of a theory.

Space of *What?*

- Category theory teaches us that one can substitute • with *any* object U and get a notion of *generalized point* $x \in_U X$ as a morphism $x : U \to X$.[3]
- This corresponds to a homomorphism $x^* : \mathrm{Cl}(X) \to \mathrm{Cl}(U)$, which in turn corresponds to a model of $\mathrm{Cl}(X)$ in $\mathrm{Cl}(U)$.
- In conclusion, X's point = $\mathrm{Cl}(X)$'s model. Classical point = classical model, generalized point = model in some Boolean algebra.
- So X is the (*classifying*) space of *models* of $\mathrm{Cl}(X)$. Stone space = space of models.

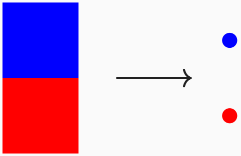
[3] This idea lies in the center of category theory as it's closely related to the mysterious *Yoneda Lemma*.

Do Logic Spatially

Question: can we recover the structure of Boolean algebra from the category Stone (or equivalently Bool)?

Let $\Sigma = \{0, 1\} \in$ Stone be the discrete space with two points. It corresponds to the theory with one propositional variable and nothing else.

we define a *clopen set* of a Stone space X to be a mapping $X \to \Sigma$.



Stone(X, Σ) has the structure of Boolean algebra. Clopen sets correspond to propositional formulas.

General Stone Duality

In general, the term *Stone Duality* might mean two different things, both involves a category of algebras $\mathcal{A}$ and a category of spaces $\mathcal{S}$.

1. Adjunction $\mathcal{S} \rightleftarrows \mathcal{A}^{\mathrm{op}}$;
2. Adjoint equivalence $\mathcal{S} \simeq \mathcal{A}^{\mathrm{op}}$.

For the purpose of this talk, we will mean the latter.

Classifying Topos

Geometric Logic

Topos corresponds to geometric logic, just like Stone space corresponds to propositional logic.

- Fix a signature Σ of sorts[4], predicate and function symbols.
- A *geometric formula* over finitely many sorted variables $\vec{x}$ is built out of $\top, \bot, \wedge, \bigvee, \exists, =$, where $\bigvee$ means *arbitrary* disjunction.
- For two formulas $\varphi(\vec{x}), \psi(\vec{x})$ over $\vec{x}$, a *geometric sequent* has the form $\varphi(\vec{x}) \vdash_{\vec{x}} \psi(\vec{x})$, understood as $\forall \vec{x}. \varphi(\vec{x}) \to \psi(\vec{x})$.
- Deduction rules are to be expected.
- A *geometric theory* is a set of geometric sequents (called *axioms* of the theory).

From now on, *theory* means geometric theory.

[4] We allow arbitrariliy many sorts, including no sort at all.

Fragments

- A theory with no sort (thus no function symbols) is a *propositional theory*. It can only have nullary predicate symbols, which are called *propositional variables*.
- The fragment with only finitary disjunction is called *coherent logic*, which is also a fragment of FOL. Coherent logic supports a form of completeness theorem that geometric logic in general doesn't have.
- Another familiar fragment is called finitary algebraic logic, or equational logic, which only features sequents of the form:

$$\top \vdash_{\vec{x}} s(\vec{x}) = t(\vec{x}).$$

Theory of Natural Numbers

As an example to show the power of $\bigvee$, consider the theory T_N with one sort N, function symbols 0 (nullary) and s (unary), axioms:

- $0 = s(x) \vdash_x \bot$ (0 is not a successor);
- $s(x) = s(y) \vdash_{x,y} x = y$ (s is injective);
- $\top \vdash_x \bigvee_{n\in\mathbb{N}} x = s^n(0)$ (everything is standard).

As we will see later, it has only one model: the standard $\mathbb{N}$. So Lowënheim-Skolem theorem fails in geometric logic.

Theory of Localic Reals

Consider the propositional theory $T_{\mathbb{R}}$ with a propositional symbol $P_{q,r}$ (thought of as the open interval $(q, r) \subset \mathbb{R}$) for every pair of rational $q, r \in \mathbb{Q}$ with axioms:

- $P_{q,r} \wedge P_{q',r'} \dashv\vdash \bigvee\{P_{s,t} \mid \max(q, q') < s < t < \min(r, r')\}$;
- $\top \vdash \bigvee\{P_{q-\epsilon,q+\epsilon} \mid q \in \mathbb{Q}, 0 < \epsilon \in \mathbb{Q}\}$.

$T_{\mathbb{R}}$ is the theory of localic reals. A model of it will be a real number. In $T_{\mathbb{R}}$, a propositional variable plays the role of a *basic open set*.

Black Magic

Here's a wicked example. Consider the propositional theory $T_{\mathbb{N}\twoheadrightarrow\mathbb{R}}$ with propositional variable $U_{n,x}$ (the open of *surjections* $f\colon \mathbb{N}\to\mathbb{R}$ that maps n to x) for each $n\in\mathbb{N}, x\in\mathbb{R}$ with axioms:

- For all $n\in\mathbb{N}$, $\top\vdash\bigvee_{x\in\mathbb{R}} U_{n,x}$ ($f(n)$ has a value);
- For all $n\in\mathbb{N}$, $x,y\in\mathbb{R}$, $U_{n,x}\wedge U_{n,y}\vdash\bot$ ($f(n)$ has only one value);
- For all $x\in\mathbb{R}$, $\top\vdash\bigvee_n U_{n,x}$ (f is surjective).

A model of it will be a surjection $\mathbb{N}\twoheadrightarrow\mathbb{R}$, so this theory has *no model* in Set, but is somehow consistent. This is essentially a form of forcing that collapses $\mathbb{R}$ into $\mathbb{N}$.

Logos

Warning: I'll use some non-standard terminologies here.

- Logos (a terminology proposed by Joyal) is the suitable structure to interpret geometric logic, just like Boolean algebra to classical propositional logic.
- The axiomatic definition of logos is called *Giraud axioms*, which is a little complicated. Here are some important parts:

A logos $\mathcal{S}X$ is a category which:

1. has finite limits;
2. has arbitrary colimits which are stable under pullback;
3. has image factorization and other nice things.

In general, think of a logos as a generalized category of sets.

Topos

A *Giraud morphism* between logoses is a functor that preserves finite limits and arbitrary colimits. A 2-morphism between Giraud morphisms is a natural transformation. They form a 2-category Logos. Here we manually define a Stone duality:

$$\text{Topos} := \text{Logos}^{\text{op}}.$$

- For any topos X, the corresponding logos is denoted as $\mathcal{S}X$, called the category of sheaves over X, the meaning of which will be clear later.
- You should think of a topos X as a *space*, and a logos $\mathcal{S}X$ as a category of generalized sets that supports interpretation of geometric logic.
- A morphism between toposes $f: X \to Y$ is traditionally called a *geometric morphism*, but you should think of it as a *continuous mapping*, so I'll call it mapping instead.-

Categorical Logic

For motivation, let's try to interpret T_N in the logos Set.

- T_N has a sort N, two symbols $0 : N$, $s : N \to N$. So a model M of T_N should contain a set N^M, an element $0^M \in N^M$, a function $s^M : N^M \to N^M$.
- Consider, for example, the geometric formula $s(x) = s(y)$ with two variables $x, y : N$. This should be interpreted as the subset: $\{(x, y) \in (N^M)^2 \mid s^M(x) = s^M(y)\} \subset (N^M)^2$.
- The sequent $s(x) = s(y) \vdash_{x,y} x = y$ poses a inclusion condition on two subsets of $(N^M)^2$.

One can easy carry out the same process in any logos, just replace "set" with "object", "function" with "morphism" and so on.

Categorical Logic

- Under the natural definition of homomorphisms between models, for every theory T and a logos $\mathcal{S}X$, one has a category $\mathrm{Mod}_T(\mathcal{S}X)$ of models of T in $\mathcal{S}X$.
- A giraud morphism $f^* : \mathcal{S}Y \to \mathcal{S}X$ preserves relevant logical structures.
- For any $M \models_{\mathcal{S}Y} T$, a 2-morphism $\alpha : f^* \Rightarrow g^*$ induces a homomorphism $\alpha^M : f^*(M) \to g^*(M)$.
- Altogether, Mod_T : Logos $\to$ CAT is a 2-functor.

Classifying Topos

Now we introduce one fundamental theorem: the functor Mod_T : Logos $\to$ CAT is *representable*.

$$\mathrm{Mod}_T \cong \text{Logos}(\mathcal{S}[T], -) : \text{Logos} \to \text{CAT}.$$

- $\mathcal{S}[T]$ plays the role of Lindenbaum algebra of T.
- The corresponding topos $[T]$ is called the *classifying topos* of T.
- By a similar argument, $[T]$ is the space of models of T.
- The theorem is commonly know as “every geometric theory has a classifying topos”.

Classifying Topos

The converse also holds, as another fundamental theorem: every topos X is a classifying topos of some geometric theory T.

The proof of it is quite difficult. It's a consequence of the following two theorems:

- Giraud theorem: any logos $\mathcal{S}X$ is equivalent to a category of sheaves on a *site* (C, J) (which, denoted as $\mathrm{Sh}(\mathsf{C}, J)$, is always a logos);
- Diaconescu theorem: $\mathrm{Sh}(\mathsf{C}, J)$ classifies continuous flat functors from (C, J).

Both theorems require *pages* of proof which we won't get into today.

Set

Set is the initial object in Logos because:

- Given any logos $\mathcal{S}X$, one can attempt to define a Giraud morphism f^* : Set $\to \mathcal{S}X$.
- f^* preserves finite limits, including terminal (singleton) $1 \in$ Set.
- Any set $S \in$ Set is the S-indexed coproduct of 1: $S \cong \coprod_{s \in S} 1$.
- f^* preserves arbitrary colimits, so $f^*(\coprod_{s \in S} 1) \cong \coprod_{s \in S} f^*(1)$ which is fixed.
- Defined that way, f^* will always be a Giraud morphism and it's the only choice.

So the corresponding topos is the terminal in Topos, thus the singleton space $\bullet$. It's funny that Set as a logos is the whole mathematical universe, but as a space is just one point.

Set Classifies the Empty Theory

Let $T_\varnothing$ be the theory with no sort, no symbol, no axiom, absolutely nothing.

- For any logos $\mathcal{S}X$, there's only one model of $T_\varnothing$ in $\mathcal{S}X$, so $\mathrm{Mod}_{T_\varnothing}(\mathcal{S}X) \cong 1$.
- We've seen that Logos(Set, $\mathcal{S}X$) $\cong 1$ as well, since Set is initial.
- So Set (or $\bullet$) classifies $T_\varnothing$.

Set Also Classifies T_N

- Consider T_N again. One can show that in any logos $\mathcal{S}X$, a model of T_N is a *natural number object* in $\mathcal{S}X$.
- Every logos has one and only one natural number object, which is also preserved by Giraud morphisms.
- So $\mathrm{Mod}_{T_N}(\mathcal{S}X) \cong 1$. Set classifies T_N as well.
- This is an example of *Morita equivalence*: two theories are Morita equivalent if they have the same classifying topos.
- One can also read it as: for any theory T, we can always add a new sort N of natural numbers (with those axioms) *for free*, it won't change the classifying topos.

Empty Space

Let $T_\perp$ be the theory with no sort, no symbol, and one axiom:

$$\top \vdash \bot.$$

- The classifying topos of $T_\perp$ is the empty space $\varnothing$.
- Its sheaf logos is the singleton category.
- I'd like to think that people living in this mathematical universe probably hate maths.

A Space without Point

- A topos X is non-empty, if $X \ncong \varnothing$.
- A topos X has no point, if $\text{Topos}(\bullet, X) \cong \varnothing$.
- Consider our wicked theory $T_{\mathbb{N}\twoheadrightarrow\mathbb{R}}$. Its classifying topos $[T_{\mathbb{N}\twoheadrightarrow\mathbb{R}}]$ has no point, since a point corresponds to a surjection $\mathbb{N} \to \mathbb{R} \in$ Set and there's no such thing.
- Nontheless $[T_{\mathbb{N}\twoheadrightarrow\mathbb{R}}]$ is non-empty, since the theory is consistent (thus not Morita equivalent to $T_\perp$).
- Takeaway: Completeness Theorem "every consistent theory has a model" translates to "every non-empty space has a point" fails in geometric logic, which makes forcing possible.

Gen Z Humor

Surjection $\mathbb{N} \twoheadrightarrow \mathbb{R}$ doesn't exi-

Object Classifier and Sheaf

Let T_O be the theory with one sort and nothing else. It has a classifying topos $[T_O]$:

$$\mathrm{Mod}_{T_O}(\mathcal{S}X) \cong \mathcal{S}X \cong \mathsf{Topos}(X, [T_O]).$$

- As a space, $[T_O]$ is weird: a point $x : \bullet \to [T_O]$ in it is a set $x \in \mathsf{Set}$. So $[T_O]$ is the *space of sets*.
- So for any topos X, a sheaf over X = an object in $\mathcal{S}X$ = a mapping $X \to [T_O]$ = a mapping from X to the space of sets.
- Intuitively, a sheaf over X is a family of sets indexed by points in X that vary continuously as a point in X moves around.
- It's precisely in this sense that a logos (category of sheaves) is a generalized category of sets.

Compare $[T_O]$ with Σ

- Recall: $\Sigma = \{0, 1\}$ is the discrete Stone space with two points, or a space of truth values.
- A clopen set U in any Stone space X is thus a family of truth values that vary continuously over X.
- It's precisely in this sense that the Boolean algebra of clopen sets $\mathrm{Cl}(X)$ is the algebra of generalized truth values.
- $[T_O]$ and Σ plays the same role in different dualities. They are both the *dualizing* object (or *schizophrenic* object as some people say).

Geometric Construction

- By definition, any classifying logos $\mathcal{S}[T]$ can be thought of as the "free logos" with a model of T.
- Consider a mapping $f\colon [T_1] \to [T_2]$. This will corresponds to a model of T_2 in $\mathcal{S}[T_1]$.
- Since $\mathcal{S}[T_1]$ is the logos freely generated from a model of T_1, two define such a model, one simply has to construct it from a model of T_1.
- The construction has to be available in any logos and preserved by Giraul morphisms.

Geometric Construction

- Define: a categorical construction is *geometric* if it's available in any logos and is preserved by Giraud morphisms.
- By definition, finite limits and arbitrary colimits are such constructions. Other examples include natural number, list object, Kuratowski-finite powerset...
- Mathematics with only geometric constructions available is called *geometric mathematics*. This amounts to doing mathematics logos-independently.

Geometric Construction

Altogether, in order to construct a mapping $f\colon [T_1] \to [T_2]$, one simply needs to perform such an argument:

- Let G be a model of T_1.
- Geometrically construct a model $f(G)$ of T_2.

Spatially, this can also be read as:

- Let G be a point in $[T_1]$.
- Geometrically construct a point $f(G)$ in $[T_2]$.

Plato's Idea

The construction can be understood in two different ways:

- For any concrete $M \models_{\mathcal{S}X} T_1$ in any logos $\mathcal{S}X$, one can substitute G with M and the argument becomes an actual construction in $\mathcal{S}X$, which gives you a $f(M) \models_{\mathcal{S}X} T_2$.
- Or, G is the generic model of T_1 in $\mathcal{S}[T_1]$, and the whole argument is an actual construction in $\mathcal{S}[T_1]$, giving you $f(G) \models_{\mathcal{S}[T_1]} T_2$.

$$\begin{array}{ccccccc} 1_{[T_1]} & \in & \mathrm{Topos}([T_1],[T_1]) & \longrightarrow & \mathrm{Topos}(X,[T_1]) & \ni & M \\ & & \big\downarrow f_* & & \big\downarrow f_* & & \\ f & \in & \mathrm{Topos}([T_1],[T_2]) & \longrightarrow & \mathrm{Topos}(X,[T_2]) & \ni & f(M) \end{array}$$

To me this resembles the Platonic notion of *Idea*.

The Real Line

Let's study the theory $T_{\mathbb{R}}$ of localic reals.

- $P_{q,r} \wedge P_{q',r'} \vdash\dashv \bigvee\{P_{s,t} \mid \max(q,q') < s < t < \min(r,r')\}$;
- $\top \vdash \bigvee\{P_{q-\epsilon,q+\epsilon} \mid q \in \mathbb{Q}, 0 < \epsilon \in \mathbb{Q}\}$.

- Read $P_{q,r}$ as open interval (q,r), $\vdash$ as inclusion, then $\wedge, \bigvee$ becomes $\cap$ and $\bigcup$. The formulas are built from these $P_{q,r}$ using $\wedge, \bigvee$ are thus opens in $\mathbb{R}$.
- The idea: these intervals form a basis of $\mathbb{R}$. The axioms are a complete set of rules obeyed by these basic opens.
- So the "Lindenbaum algebra" of $T_{\mathbb{R}}$ is just the frame of opens $O(\mathbb{R})$.

The Real Line

- Quesiton: what's a *model* of $T_{\mathbb{R}}$?
- Suppose $M \models_{\mathsf{Set}} T_{\mathbb{R}}$. It sends every formula U (thus an open $U \in O(\mathbb{R})$) to a truth value $\top, \bot$. So M is a function $O(\mathbb{R}) \to 2$.
- By the definition of model, M preserves order, $\wedge$, $\bigvee$. By topology, this corresponds exactly to a *point* $m \in \mathbb{R}$: $M(U) = \top$ iff $m \in U$.
- The classifying topos $[T_{\mathbb{R}}]$ is then the space of reals, which is just the real line $\mathbb{R}$. So we define $\mathbb{R}$ to be the topos $[T_{\mathbb{R}}]$.

Square Function

Let's define the square function $x \mapsto x^2 : \mathbb{R} \to \mathbb{R}$ with our framework.

- Take a model $x \models T_{\mathbb{R}}$. It has the data of all the rational open intervals (s, t) it belongs to.
- To define $x^2 \models T_{\mathbb{R}}$, we need to define what (q, r) it belongs to. Well, $x^2 \in (q, r)$ iff there's an $(s, t) \ni x$ such that $(s^2, t^2) \cup (t^2, s^2) \subseteq (q, r)$.
- Once we check that these conditions indeed define a model of $T_{\mathbb{R}}$, we're done.
- As a mapping beteen toposes, $x \mapsto x^2$ is automatically continuous.
- Moral of the story: by restricting to geometric mathematics, all definable functions are automatically continuous.

Take Away

Summary of topos theory:

- To present a space, simply write down the geometric theory of its points.
- To define a continuous mapping between spaces $f\colon [T_1] \to [T_2]$, take a model of T_1 and geometrically construct a model of T_2.
- A point in a point-free space X is a generic filter, as set theorists would say. It doesn't exist in Set, but it does in $\mathcal{S}X$.
- Restricted to geometric mathematics, one can deal with point-free spaces pointwise. (This is still being researched actively, mainly by Steven Vickers.)