# Exercises 4

Gabriel Goldberg

July 21, 2022

Work in ZF unless otherwise noted.

- Show that every almost supercompact cardinal is rank reflecting.
- Show in ZFC that every supercompact cardinal is $\Sigma_2$-reflecting and a limit of $\Sigma_2$-reflecting cardinals.
- Show that every $\Sigma_2$-reflecting cardinal is rank reflecting.
- Show in ZFC that every almost supercompact cardinal is either supercompact or a limit of supercompacts.
- Question: in ZFC, must rank reflecting cardinals be $\Sigma_2$-reflecting?
- Prove the dichotomy theorem mentioned in class:

  **Theorem.** *Suppose $\lambda$ is the least rank Berkeley cardinal.*

  - *A cardinal $\kappa \leq \lambda$ is almost supercompact iff it is supercompact or a limit of supercompacts.*
  - *A cardinal $\kappa \geq \lambda$ is almost supercompact iff it is rank reflecting.*

  - The first bullet is similar to the fact in ZFC that every almost supercompact cardinal is either supercompact or a limit of supercompacts.
  - The second is similar to the proof from lecture that a rank Berkeley implies a proper class of almost supercompacts.
  - See "Choiceless cardinals and the continuum problem" if you get stuck.

- Prove that for any ordinal $\alpha$, the least rank reflecting cardinal above $\alpha$ has countable cofinality.
- Show that if $F$ is a $\kappa$-complete filter on an ordinal and $\kappa$ is not the surjective image of $V_\alpha$ for any $\alpha < \kappa$, then for any family $\langle A_x \rangle_{x \in V_\alpha} \subseteq F$, $\bigcap_{x \in V_\alpha} A_x \in F$.
- Show that if $\kappa$ is rank reflecting, then for $\alpha < \kappa$, there is no surjection from $V_\alpha$ to $\kappa$.
- Show that if the wellordered collection lemma holds at $\lambda$, then $\lambda^+$ is regular.
- Show that in fact, for any set $X$ such that $X$ surjects onto $X \times X$ and $\lambda \leq X$, the supremum of all ordinals $\eta$ that are the image of $X$ under a surjective function has cofinality at least $\lambda^+$.

- Show that for any ordinal $\gamma$, for all sufficiently large $\alpha$, the least Berkeley cardinal $\delta \geq \alpha$ has cofinality greater than $\gamma$.
  - Recall that $\delta$ is *Berkeley* if for all structures $M$ in a language of size less than $\delta$, if $|M| \geq \delta$, then there is a nontrivial elementary embedding $j : M \to M$.
  - Note that the least Berkeley above $\alpha$ can be characterized as the least ordinal $\delta$ such that for all transitive sets $M$ containing $\delta$, there is an elementary embedding $j : M \to M$ with $\mathrm{crit}(j) > \alpha$.
  - Woodin gave a simple proof that Berkeley cardinals are incompatible with AC: fix $(\gamma_\xi)_{\xi < \mathrm{cf}(\delta)}$ cofinal in the least Berkeley cardinal $\delta$, and choose $M_\xi$ such that there is no $j : M_\xi \to M_\xi$ with $\mathrm{crit}(j) < \gamma_\xi$; Berkeleyness yields an embedding from $\left(\bigcup_{\xi < \mathrm{cf}(\delta)} M_\xi, \langle M_\xi \rangle_{\xi < \mathrm{cf}(\delta)}\right)$ to itself, which easily gives a contradiction.
  - The exercise can be solved by using the wellordered collection lemma to simulate Woodin's proof.
- Question: can one refute Berkeley cardinals outright in ZF?