Potentialism in the philosophy and foundations of mathematics

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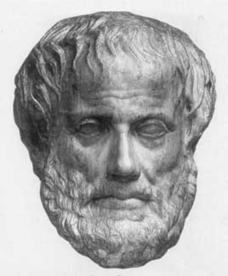
Aristotle's notion of potential infinity



"For generally the infinite is as follows: there is *always* another and another to be taken. And the thing taken will always be finite, but always different." (*Physics*, 206a27-29).

A stick s is **infinitely divisible**:

(1) Necessarily, for any proper part x of s, possibly x has a proper part

However, Aristotle denies that s is, or even could be, **infinitely divided**:

(2) ~~Necessarily,~~ for any proper part x of s, ~~possibly~~ x has a proper part

The plan

1. potentialism in the history of mathematics and philosophy (L. & Shapiro, 2019)
2. potentialism and modal logic: mirroring theorems (L., 2010)
3. potentialist mathematics: *modal* set theory (L., 2013)
4. potentialism and critical plural logic
5. potentialist mathematics *without modality*: sets and extensional abstraction
6. taming intensionality

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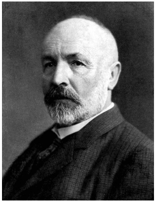
A *very* brief historical overview

Aristotle's conception of infinity dominated for more than two millenia.

Gauss wrote in 1831:

> *I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics.*

Then came Cantor . . .



- The actual or completed infinite is permissible.
- There is an unbounded sequence of infinite numbers.

Cantor and Zermelo on incompletability

[We must] distinguish two kinds of multiplicities [. . .] For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call absolutely infinite or inconsistent multiplicities. [. . .] If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a consistent multiplicity or a 'set'. (1899 letter to Dedekind)

the transfinite number series [. . .] reaches no true completion in its unrestricted advance, but possesses only relative stopping-points, just those 'boundary numbers' (Zermelo, 1930)

Arithmetical vs. set-theoretic potentialism

Aristotle is a potentialist about arithmetic:

$$\Box\forall m\Diamond\exists n\,\textsc{Succ}(m, n)$$
$$\neg\Diamond\forall m\exists n\,\textsc{Succ}(m, n)$$

Cantor and Zermelo appear to be potentialists about set-theory:

$$\Box\forall xx\Diamond\exists y\,\textsc{Set}(xx, y)$$
$$\neg\Diamond\forall xx\exists y\,\textsc{Set}(xx, y)$$

Comparison

- *Thresholds*: While arithmetical potentialism is concerned with ω, set-theoretic potentialism is concerned with Ω.
- *Paradox*: While completion of the natural numbers is consistent, completion of the sets is not.

Potentialism in constructive mathematics

A **Brouwerian free choice sequence** is a potentially infinite sequence.

Predicativists too regard certain domains as merely potential.

> *"inexhaustibility" is essential to the infinite. (Weyl, 1918, 23)*

> *... we can never speak sensibly (in the predicative conception) of the "totality" of all sets as a "completed totality" but only as a potential totality whose full content is never fully grasped but only realized in stages. (Feferman, 1964, 2)*

Task 2: Potentialism & modal logic: mirroring theorems

Ordinary mathematics is done in a non-modal language $\mathcal{L}$, whereas our analysis of potentialism is in a corresponding modal language $\mathcal{L}^\Diamond$.

What is the correct logic for reasoning in $\mathcal{L}$ about a merely potential domain? The answer will depend on

1. the translation Tr from $\mathcal{L}$ to $\mathcal{L}^\Diamond$
2. the modal theory $T^\Diamond$ of the potential domain

$$\begin{tikzcd} \mathcal{L} \arrow[r, bend left, "Tr"] & \mathcal{L}^\Diamond \\ ? & T^\Diamond \end{tikzcd}$$

The Gödel translation

The non-trivial clauses of the translation G are:

$$\begin{aligned} \varphi &\mapsto \Box\varphi \qquad \text{for } \varphi \text{ atomic} \\ \neg\varphi &\mapsto \Box\neg\varphi^G \\ \varphi \to \psi &\mapsto \Box(\varphi^G \to \psi^G) \\ \forall x\, \varphi &\mapsto \Box\forall x\, \varphi^G \end{aligned}$$

Theorem (Mirroring via the Gödel translation)

Let $\vdash_{int}$ be intuitionistic first-order deducibility. Let $\vdash_{S4}$ be deducibility in classical first-order logic plus S4. Then we have:

$$\varphi_1, \ldots, \varphi_n \vdash_{int} \psi \quad \textit{iff} \quad \varphi_1^G, \ldots, \varphi_n^G \vdash_{S4} \psi^G.$$

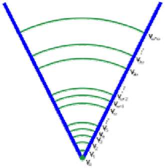
The Gödel translation is *totally unsuited* to explicate potentialism:

$$\forall m \exists n \, \textsc{Succ}(m, n)$$
$$\Box \forall m \exists n \, \textsc{Succ}(m, n)$$

(Parsons, 1983, 321-22) suggests composing the double negation translation with the Gödel translation:

$$\forall m \exists n \, \textsc{Succ}(m, n)$$
$$\Box \forall m \Box \neg \Box \forall n \Box \neg \, \textsc{Succ}(m, n)$$
$$\text{i.e., } \Box \forall m \Box \Diamond \exists n \Diamond \, \textsc{Succ}(m, n)$$

The potentialist translation (L. 2010)



- Translate '$\forall$' and '$\exists$' as'$\Box\forall$' and '$\Diamond\exists$', respectively.
- Claim: the translation $\varphi \mapsto \varphi^{\Diamond}$ provides the desired translation.
- E.g., '$\forall m \exists n \textsc{Succ}(m), n)$' translates as '$\Box\forall m \Diamond\exists n\, \textsc{Succ}(m, n)$'

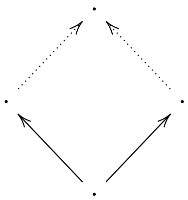
Stability of the atomic predicates:

$$P(\mathbf{u}) \to \Box P(\mathbf{u}) \qquad (\textsc{Stb}^{+}\text{-}P)$$

$$\neg P(\mathbf{u}) \to \Box \neg P(\mathbf{u}) \qquad (\textsc{Stb}^{-}\text{-}P)$$

What is the right modal logic? At least S4.

It is plausible to assume that the extensions are "convergent":



This licences the adoption of one more axiom:

$$\Diamond\Box\varphi \to \Box\Diamond\varphi \qquad \text{(G)}$$

So we adopt S4.2 = S4 + G.

Theorem (First-order potentialist mirroring (L. 2010))

Let $\vdash^{\Diamond}$ be provability by $\vdash_{FOL}$, S4.2, and axioms stating that every atomic predicate is stable, but with no higher-order comprehension. Then we have:

$$\varphi_1, \ldots, \varphi_n \vdash_{FOL} \psi \quad \textit{iff} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

This refutes an objection due to Cantor:

> *every potential infinite, if it is to be applicable in a rigorous mathematical way, presupposes an actual infinite. [Cantor, 1887, pp. 410–411]*

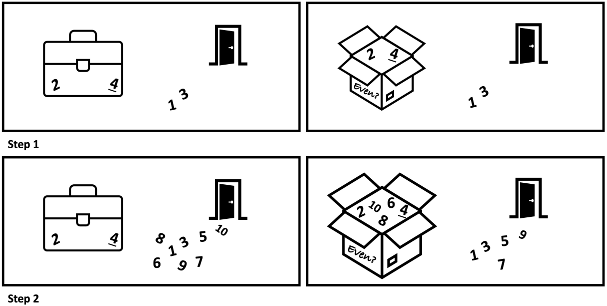
Theorem (Intuitionistic potentialist mirroring (L. & Shapiro 2019))

For any formulas $\varphi_1, \ldots, \varphi_n$, and ψ of $\mathcal{L}$, we have:

$$\varphi_1, \ldots, \varphi_n \vdash_{int} \psi \quad \textit{iff} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash_{int}^{\Diamond} \psi^{\Diamond}.$$

Task 3: potentialist mathematics—modal set theory

combinatorial sets: logical classes:



Plurals and the definition of sets

Q: Can we define a set of all and only the objects that satisfy $\varphi(x)$:

$$\exists y \forall x (x \in y \leftrightarrow \varphi(x))$$

Suppose the desired set is **specified intensionally**—which we explicate by translating **Q** into $\mathcal{L}^{\Diamond}$:

$$\Diamond \exists y \, \Box \forall x (x \in y \leftrightarrow \varphi^{\Diamond}(x)) \tag{S-Int$^{\Diamond}$}$$

This is dangerous: the desired set is specified in a potentially shifty way!

When the target is **specified extensionally**, the definition is permissible:

$$\forall u (u \prec xx \leftrightarrow \varphi^{\Diamond}(u))$$
$$\Diamond \exists y \, \Box \forall x (x \in y \leftrightarrow x \prec xx) \tag{S-Ext$^{\Diamond}$}$$

Modal set theory

Perfectly naive set comprehension is permissible—so long as the target is specified extensionally—and thus in a non-shifty manner:

$$\Box\forall xx\, \Diamond\exists y\, \textsc{Set}(xx, y) \qquad \text{(Collapse}^{\Diamond}\text{)}$$

But a plurality can only draw its members from a single world:

$$\Box\exists yy \;\; \forall x(x \prec yy \leftrightarrow \varphi(x)) \qquad \text{(P-Comp)}$$

One way to proceed is to develop a modal set theory in $\mathcal{L}^{\Diamond}$ and utilize the mirroring theorem (Linnebo, 2013), (Studd, 2013).

Task 4: Potentialism and critical plural logic

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, bend left, "\Diamond"] & & \mathcal{L}^{\Diamond} \\
? & & \text{modal logic of plurals}
\end{tikzcd}$$

Traditional plural logic says there is a universal plurality: $\exists xx \forall y\, y \prec xx$. This translates as:

$$\Diamond \exists xx \Box \forall y (y \prec xx) \tag{1}$$

But potentialism requires the opposite, namely:

$$\Box \forall xx\, \Diamond \exists y (y \not\prec xx) \tag{2}$$

So we need a **critical plural logic** that is validated in $\mathcal{L}$ under the potentialist translation.

The modal logic of plurals

We start with **traditional non-modal plural logic**: PFO+.

Concerning **the interaction of modals and plurals**, our *key idea* is that a plurality is tracked across possible worlds in terms of its members, so that it has its members by necessity.

E.g., the one-of relation $\prec$ is stable:

$$x \prec yy \to \Box x \prec yy \qquad (\textsc{Stb}\prec)$$

$$x \not\prec yy \to \Box x \not\prec yy \qquad (\textsc{Stb}\not\prec)$$

Let us call the resulting modal logic of plurals **MPFO+**.

Towards a critical plural logic

If not traditional plural logic, what plural logic *is* validated in $\mathcal{L}$?

$$
\begin{tikzcd}
\mathcal{L} \arrow[rr, bend left, "\Diamond"] & & \mathcal{L}^{\Diamond} \\
? & & \mathit{MPFO}+
\end{tikzcd}
$$

Let **Basic Plural Logic (BPL)** be the plural logic obtained from a complete axiomatization of first-order logic by adding:

- the usual axioms and rules governing the plural quantifiers, plural indiscernibility;
- that every object yields a singleton plurality;
- that any two pluralities have a union plurality;
- that any xx and condition $\varphi(x)$ there is a plurality of those members of xx that satisfy $\varphi(x)$

Let $\vdash^{\mathrm{BPL}}$ be provability in $\mathcal{L}$ using BPL and $\vdash^{\mathrm{MPFO+}}$ be provability in $\mathcal{L}^{\Diamond}$ using MPFO$+$ and the stability axioms.

Theorem (Plural mirroring)

We have:

$$\varphi_1, \ldots, \varphi_n \vdash^{BPL} \psi \quad \textit{iff} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{MPFO+} \psi^{\Diamond}$$

Upshot: potentialists who rely on the potentialist translation $\varphi \mapsto \varphi^{\Diamond}$ and MPFO$+$ in $\mathcal{L}^{\Diamond}$ are *thereby* entitled to BPL.

Of course, more in, more out ...

Can we define **a translation in the reverse direction**?

$$\begin{tikzcd} \mathcal{L} \arrow[r, bend left, "\Diamond"] & \mathcal{L}^{\Diamond} \arrow[l, bend left, "?"] \end{tikzcd}$$

Idea: A possible world can be represented by the plurality of objects in its domain. This works because:

- all atomic predicates are stable
- the accessibility $\leq$ between possible worlds is convergent

This enables us to prove that two slightly tweaked systems, BPL* and MPFO*, are **definitionally equivalent**—a *very* tight form of equivalence.

Task 5: potentialist mathematics *without modality*

Set-theoretic potentialism is based on the idea that, whenever there are some objects xx, these can be used to define a set $\{xx\}$.

This suggests a plural analogue of Frege's Basic Law V:

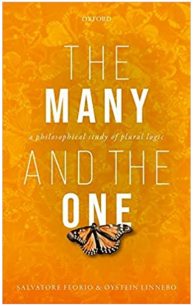
(i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$

Since every set is generated in this way, we add an induction scheme:

(ii) Suppose that every urelement is φ and that, for any xx each of which is φ, $\{xx\}$ too is φ. Then everything is φ.

Theorem (Florio & Linnebo, 2020, 2021)

1. *BPL and the principles (i) and (ii) prove: Extensionality, Foundation, Empty Set (if we admit an empty plurality), Pair, Separation.*
2. *The other axioms of ZFC—Union, Replacement, Powerset, Infinity, AC—follow from a natural extension of BPL, i.e. critical plural logic.*



Abstractionism and bad company

Frege and neo-Fregeans are attracted to Hume's Principle:

$$\#F = \#G \leftrightarrow F \approx G \tag{HP}$$

The bad company problem: Permissible abstraction principles

$$\S\alpha = \S\beta \leftrightarrow \alpha \sim \beta \tag{AP}$$

are mixed in among impermissible ones (e.g. Basic Law V).

Extensional abstraction using critical plural logic

Solution: In Basic (or Critical) Plural Logic, it is *always* permissible to abstract on pluralities

$$\S xx = \S yy \leftrightarrow xx \sim yy \tag{$\mathrm{AP_{ext}}$}$$

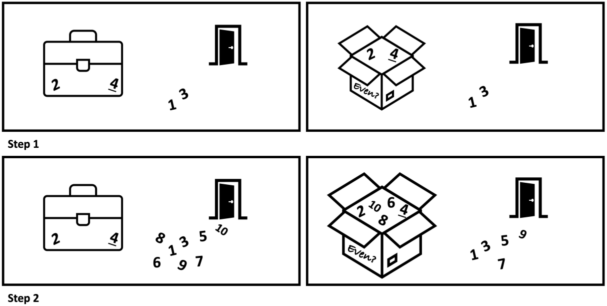
provided that $\sim$ quantifies just over xx and yy (cardinals, sets, etc.).

Heuristic: We can freely abstract on "old" objects to obtain "new" ones, provided the equivalence $\sim$ is concerned solely with the "old" objects.

Task 6: taming intensionality

Recall our distinction between extensional and intensional collections:

combinatorial sets *logical classes*



How to generalize over an "open box"

When the domain is an extensional collection, we can give an **instance-based** explanation:

- $\forall x \, \varphi(x)$ because $\varphi(a), \varphi(b), \ldots$.

When the domain is an intensional but not an extensional collection, we need a **non-instance-based** explanation:

- That all whales are mammals can be explained in terms of the properties in question, with no mention of any particular whale.
- That every set s has a singleton $\{s\}$ can be explained in terms of the operation $s \mapsto \{s\}$, with no mention of any particular set.

Intrinsic truthmaking and (semi-)intuitionistic logic

"Generality Explained" (Linnebo, 2022) develops an account of intrinsic truthmaking:

- $s \Vdash \varphi$ as "the (perhaps partial) state of the world s suffices to explain φ's truth"
- s can support non-instance-based explanations of $\forall x\, \psi(x)$
- the natural logic of $\Vdash$ is **semi**-intuitionistic:
  - classical when quantifiers are restricted to a plurality (since then instance-based explanations are available)
  - only intuitionistic for unrestricted quantification

A theory of well-behaved "boxes"

For the definition of a collection to be permissible, the target must be specified in a **stable**—or non-shifty—manner; i.e., the definition must not be disrupted by the generation of further objects.

- For combinatorial sets, definitional stability is ensured by using pluralities: $xx \mapsto \{xx\}$.
- For logical classes, the defining condition must be stable (Poincaré, 1909); i.e. $\forall a$ at any stage s:

$$s \text{ “says” } \varphi(a) \quad \text{iff} \quad \forall t \geq s : t \text{ “says” } \varphi(a)$$

How to ensure that the defining condition is stable?

- Restrict to **available objects**: classical logic, but φ is subject to Russell's Vicious Circle Principle (L. & Shapiro, 2021).
- Use only **available information**: "says" as intrinsic truthmaking, so semi-intuitionistic logic, but require $\forall x(\varphi(x) \vee \neg\varphi(x))$, cf. Feferman.

Summary

Potentialist ideas

- have a rich history
- can be developed rigorously using modal logic
- can be connected in a simple and natural way with the non-modal language of ordinary math
- have substantial explanatory power in connection with combinatorial sets
- can even be developed without modality using critical plural logic
- can be extended so as to tame intensionality, e.g. logical classes

Ewald, W. (1996).
From Kant to Hilbert: A Source Book in the Foundations of Mathematics, volume 2.
Oxford University Press, Oxford.

Feferman, S.
Is the continuum hypothesis a definite mathematical problem?
Unpublished manuscript dated 2011.

Feferman, S. (1964).
Systems of predicative analysis.
Journal of Symbolic Logic, 29(1):1–30.

Florio, S. and Linnebo, Ø. (forthcoming).
The Many and the One: A Philosophical Study.
Oxford University Press, Oxford.

Linnebo, Ø. (2010).
Pluralities and sets.
Journal of Philosophy, 107(3):144–164.

Linnebo, Ø. (2013).

The potential hierarchy of sets.
Review of Symbolic Logic, 6(2):205–228.

Linnebo, Ø. (2017).
Philosophy of Mathematics.
Princeton University Press, Princeton, NJ.

Linnebo, Ø. (2022).
Generality explained.
Journal of Philosophy, 119(7):349–379.

Linnebo, Ø. and Shapiro, S. (2019).
Actual and potential infinity.
Noûs, 53(1):160–191.

Linnebo, Ø. and Shapiro, S. (202x).
Predicativism as a form of potentialism.
Review of Symbolic Logic, pages 1–32.

Parsons, C. (1983).
Mathematics in Philosophy.
Cornell University Press, Ithaca, NY.

Poincaré, H. (1909).
La logique de l'infini.
Revue de Métaphysique et de Morale, 17(4):461–482.

Studd, J. (2013).
The iterative conception of set: A (bi-)modal axiomatisation.
Journal of Philosophical Logic, 42(5):697–725.

Weyl, H. (1918).
Das Kontinuum.
Verlag von Veit & Comp, Leipzig.
Translated as *The Continuum* by S. Pollard and T. Bole, Dover, 1994.

Zermelo, E. (1930).
Über Grenzzahlen und Mengenbereiche.
Fundamenta Mathematicae, 16:29–47.
Translated in (Ewald, 1996).