Borel Chain Conditions on Borel Partial Orderings

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Keywords

1. Categorization of posets
2. Borel combinatorics on posets

Back ground, part. I: Partially ordered sets and Boolean algebras

Definition

A set P equipped with a binary relation $\leq$ is said to be a poset if $\leq$ is transitive, reflexive and antisymmetric.

Definition

A Boolean algebra is algebraic structure $(B, 0, 1, \vee, \wedge, -)$ satisfying the usual laws of fields of sets.
A Boolean algebra B is considered as poset with the ordering $x \leq y$ iff $x \vee y = y$.
A Boolean algebra is complete if every subset X of B has the least upper bound $\bigvee X$ or equivalently greatest lower bound $\bigwedge$ relative to the ordering $\leq$ of B.

Back ground, part. I: The roles of posets in set theory

1. Captures the structure of posets as forcing notions and complete Boolean algebras supporting strictly positive sub-mesures and measures,
2. Used in the theory of Forcing to distinguish between different forcing extensions,
3. Used in the theory of Forcing Axioms to calibrate their strengths.

Back ground, part. I: The statement of the general problem

Question

How to categorize posets?
Which internal properties of a given poset P have influence to the properties of its completion and the properties of its forcing extensions?
Which internal properties of a given complete Boolean algebra B determine the possibilities for the existence of various kind of strictly positive sub-measures and measures on B?

Back ground, part. I: The combinatorics of posets

Definition

Let P be a poset. A pair of different elements $p \neq q \in P$ is said to be compatible if there is an r such that $r \leq q$ and $r \leq p$. They are incompatible if they are not compatible.

Definition

A subset $A \subset P$ is said to be:

1. an antichain if every two different elements from A are incompatible.
2. n-linked if for every n elements, there is an r that is $\leq$ to all of them.
3. centred if it is n-linked for every n.

Definition

A σ-complete Boolean algebra B is (ω, ω)-*distributive* whenever for every sequence a_{mn} $(m, n < \omega)$ of nonzero elements of B, we have that

$$\bigwedge_{m<\omega} \bigvee_{n<\omega} a_{mn} = \bigvee_{f\in\omega^\omega} \bigwedge_{m<\omega} a_{mf(m)}.$$

Definition

A σ-complete Boolean algebra B is *weakly* (ω, ω)-*distributive*, or simply *weakly distributive*, whenever for every sequence a_{mn} $(m, n < \omega)$ of nonzero elements of B, we have that

$$\bigwedge_{m<\omega} \bigvee_{n<\omega} a_{mn} = \bigvee_{f\in\omega^\omega} \bigwedge_{m<\omega} \bigvee_{n<f(m)} a_{mn}.$$

Back ground, part. I: Von Neumann's problem

Definition

A poset or a Boolean algebra is said to satisfy the countable chain condition (ccc, in short) if it includes no uncountable anti-chains.

By observing that every Boolean algebra supporting strictly positive σ-additive measure is necessarily ccc and weakly distributive, Von Neumann has asked the following question:

Question (Von Neumann 1937)

Is it true that every weakly distributive ccc complete Boolean algebra supports a strictly positive σ-additive measure?

Back ground, part. I: Von Neumann's problem

Definition

Given a Boolean algebra B, a function $f\colon P \to \mathbb{R}$ is:

1. Strictly positive if $f(p) > 0$ for all $p \in P$.
2. Exhaustive if for every countable antichain $A = \{a_n\}$, $lim_{n\to\infty} P(a_n) = 0$.
3. A submeasure if for every pair $p \subset q$ we have $f(p) \leq f(q)$ and for every pair of incompatible $p, q \in P$, $f(p) + f(q) \leq f(p \vee q)$.

Back ground, part. I: ...and the solutions

Theorem (Maharam 1947, Jech 1966, Tennenbaum 1965)

It is consistent with ZFC that there is a complete ,ccc, distributive and non-atomic Boolean algebra (the Souslin algebra). Such algebra supports no strictly positive continuous submeasure and so, in particular, it supports no finitely additive strictly positive measure.

Back ground, part. I: The hierachy of chain conditions

Definition

Given a poset, it is said to be:

1. σ-finite chain condition (σ-fcc) if it is a union of countably many subsets which includes no infinite antichains.
2. σ-bounded chain condition (σ-bcc) if it is a union of countably many subsets in which there is an upper bound for the size of antichains.
3. σ-n-linked if it is an union of countably many n-linked subsets.
4. σ-linked if it is σ-2-linked.
5. σ-centred if it is a union of countable many centred subsets.

Back ground, part. I: The hierarchy of chain conditions

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Theorem (Many authors)

This hierarchy is strict. Moreover, there is a poset that is σ-n-linked for every n but fails to be σ-centred.

σ-n-chain condition is σ-2-chain condition

Definition

A poset is said to satisfy the σ-n-chain condition(σ-n-cc) if it is a union of countably many subsets which includes no antichains of size $\geq n$.
It is said to satisfy the Borel σ-n-chain condition(Borel σ-n-cc) if these subsets can be chosen to be Borel.

Theorem (Galvin and Hajnal 1977)

Every σ-n-cc poset is σ-linked.

Back ground, part. I: Maharam's version of Von Neumann's problem

Theorem (Balcar, Jech and Pazák 2003)

Assume the P-ideal dichotomy. Let B be a complete Boolean algebra. The following are equivalent:

1. *B is ccc and σ-weakly distributive.*
2. *There is a strictly positive exhaustive submeasure on B.*

Theorem (Todorcevic 2004)

Let B be a complete Boolean algebra. The following are equivalent:

1. *B satisfies σ-finite chain condition and is σ-weakly distributive.*
2. *There is a strictly positive exhaustive submeasure on B.*

Back ground, part. II: The descriptive combinatorics

Definition

A Polish space is a separable completely metrizable space. Given a Polish space X, a Borel set is an element of the σ-algebra generated by its open sets.

Fact

Every Polish space is a continuous surjective image of the Baire space ω^ω.

Back ground, part. II: The descriptive combinatorics

Definition

A Borel graph is graph $G = (V, E)$ such that V is a Polish space and E is a Borel subset of the product space V^2.

Definition

A Borel poset is a poset $(P, \leq)$ such that P is a Polish space and $\leq$ is a Borel subset of the product space P^2.

Back ground, part. II: The descriptive combinatorics

Definition

The Borel chromatic number $\chi_B(G)$ of a Borel graph $G = (V, E)$ is the smallest cardinality κ such that there is a Polish space X and an edge preserving Borel mapping from G to $(X, \emptyset)$.

Theorem (Kechris, Solecki and Todorcevic 1999)

There is a Borel graph G_0 so that for any Borel graph G, exactly one of the following happens:

1. $\chi_B(G) \leq \aleph_0$*, or*
2. *There is a continuous edge preserving mapping from G_0 to G.*

Theorem (Harrington, Marker and Shelah 1988)

Let $(P, \leq)$ be a Borel poset. If it cannot be written as a countable union of Borel chains, then it includes a perfect subset of pairwise incomparable elements.

The hierachy of Borel chain conditions

Definition

Given a poset, it is said to be:

1. Borel σ-finite chain condition (σ-fcc) if it is a union of countably many Borel subsets which includes no infinite antichains.
2. Borel σ-bounded chain condition (σ-bcc) if it is a union of countably many Borel subsets in which there is an upper bound for the size of antichains.
3. Borel σ-n-linked if it is an union of countably many Borel n-linked subsets.
4. Borel σ-linked if it is Borel σ-2-linked.
5. Borel σ-centred if it is a union of countable many Borel centred subsets.

A first observation

Definition

Given a topological space X, we define the Todorcevic Order $T(X)$ over X to be the poset consists with compact subsets of X with only finitely many limit points, ordered by reverse inclusion that preserves non-limit points.

Fact

When X is countable, $T(X)$ can be regarded as a subspace of 2^X equipped with the usual product topology.

A first observation

Theorem (Todorcevic-X.2020)

Suppose for a Borel definable topological space X there is a collection of analytic subsets $C = \{X_t : t \in 2^{<\omega}\}$ such that:

1. *If $t \sqsubseteq s$ then $X_s \subset X_t$.*
2. *For each $b \in 2^\omega$, $\cap_{n<\omega} X_{b|n}$ is a singleton. For each branch b, call the only element in this singleton x_b.*
3. *For any sequence $\{b_k \in 2^\omega\}_{k<\omega}$ and $b \in 2^\omega$ such that $\lim_{k\to\infty} |b \vee b_k| = \omega$ and $b(|b \vee b_k| + 1) = 0$ for all k, then $x_{b_k} \to x_b$ in X for any $x_{b_k} \in \cap_{n<\omega} X_{b_k|n}$ and $x_b \in \cap_{n<\omega} X_{b|n}$. (Here $b_0 \vee b_1$ denote the maximum node contained in both b_0 and b_1 for two branches b_0 and b_1 of the tree $2^{<\omega}$).*

Then $T(X)$ is not Borel σ-fcc.

Corollary (Todorcevic-X.2020)

There is a poset that is σ-fcc but not Borel σ-fcc.

Sketch of the proof of the theorem

Proof.

Given a collection $\{X_t : t \in 2^{<\omega}\}$ as stated in the theorem, Let $P = \{\langle r_0, r_1, ..., r_n\rangle$: for all $0 < k \leq n$, $r_k \in 2^{<\omega}$, there is an strictly increasing sequence $\{h_k\}_{0<k\leq n}$ of natural numbers so that $|r_n| > h_n, r_0(h_k) = 0$ for all $0 < k \leq n$ and $r_k \sqsupseteq r_0|_{(h_k-1)} \frown 1\}$. Order it by $p_0 < p_1$ if p_0 coordinate-wisely extends a $p' \in P$ that end extends p_1 as a sequence.

$\square$

Sketch of the proof of the theorem

continued...

Then any generic ultrafilter G of P gives a sequence as in the condition (3) of the theorem and thus gives a convergent sequence S_G with first coordinate being its limit.
If $P = \bigcup_n P_n$, by Shoenfield's absoluteness theorem, there is a $p \in P$ and an integer n so that $S_G \in P_n$ whenever $p \in G$.
Starting with a generic ultrafilter G_0 containing p, we can construct a sequence of generic ultrafilters G_n extending containing p, such that the first coordinate (the limit) of S_{G_n} is the n'th coordinate of S_{G_0} and the next n coordinates are the same with the corresponding coordinates of S_{G_0}.
Then S_{G_n} is an infinite antichain in $T(X)$. $\square$

Recall the hierachy of Borel chain conditions

Definition

Given a poset, it is said to be:

1. Borel σ-finite chain condition (σ-fcc) if it is a union of countably many Borel subsets which includes no infinite antichains.
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3. Borel σ-n-linked if it is an union of countably many Borel n-linked subsets.
4. Borel σ-linked if it is Borel σ-2-linked.
5. Borel σ-centred if it is a union of countable many Borel centred subsets.

...and it is distinguished

Theorem (X.)

This hierarchy is strict. Moreover, there is a Borel poset that is Borel σ-n-linked for every n but fails to be Borel σ-centred.

σ-n-chain condition is σ-2-chain condition

Definition

A poset is said to satisfy the Borel σ-n-chain condition(σ-n-cc) if it is a union of countably many Borel subsets which includes no antichains of size $\geq n$.

It turns out that this result admits its Borel version.

Theorem (X.)

Every Borel σ-n-cc poset that has Borel incompatibility is Borel σ-linked.

Posets $\mathbb{D}(G)$ associated to hypergraphs G

Definition

A (non-directed simple) hypergraph on a set X is the pair $G = (X, E)$ where $E \subset \bigcup_{1<n<\omega} X^n$.
G is a Borel hypergraph when X is a Polish space and E is Borel with the induced topology.

Definition

Let $G = (X, E)$ be a hypergraph. The poset $\mathbb{D}(G)$ consists with the finite anticliques of G, ordered by reverse inclusion.

Fact

When G is a Borel hypergraph, then $\mathbb{D}(G)$ is naturally a Borel poset.

A family of hypergraphs inspired by the graph G_0

For each tree T, fix a dense subset $D_T \subset T$ that intersects each level exactly once. Consider the following hypergraphs:

Definition

1. $G_0(n)$ is the graph defined on the branches of the tree $T = n^\omega$. The edges are of the form $\{d \frown \{i\} \frown t, d \frown \{j\} \frown t\}$ for some $d \in D_T$, $t \in [T]$ and $0 \leq i \neq j < n$.
2. $G_0(< \omega)$ is the graph defined on the branches of tree $T = [\bigcup_{n<\omega} n^n]$. The edges are of the form $\{d \frown \{i\} \frown t, d \frown \{j\} \frown t\}$ for some $d \in D_T$, $t \in [\bigcup_{n>|d|} n^n]$ and $0 \leq i \neq j \leq |d|$.
3. $G_0'(< \omega)$ is the hypergraph defined on the branches of tree $T = [\bigcup_{n<\omega} n^n]$. The edges are of the form $\{d \frown \{i\} \frown t\}_{0 \leq i < |d|}$ for some $d \in D_T$, and $t \in [\bigcup_{n>|d|} n^n]$.

Distinguishing the hierarchy of Borel chain conditions

Theorem (X.)

1. $\mathbb{D}(G_0(<\omega))$ *is Borel* σ*-fcc but is not Borel* σ*-bcc.*
2. $\mathbb{D}(G_0(2))$ *is Borel* σ*-bcc but is not* σ*-n-linked for any n.*
3. *For every* $n > 2$, $\mathbb{D}(G_0(n))$ *is Borel* σ*-*$(n-1)$*-linked but is not Borel* σ*-n-linked.*
4. $\mathbb{D}(G_0'(<\omega))$ *is Borel* σ*-n-linked for every n but is not Borel* σ*-centred.*

Sketch of the proof, part.1

Proof.

The fact that $\mathbb{D}(G_0(2))$ cannot be Borel σ-linked follows from the classical result that the Borel chromatic number of $G_0(2)$ is uncountable:
Every non-meager Borel set is comeager in some open subset, and every open subset includes an edge. $\square$

Sketch of the proof, part.2

Proof.

Use the following partition:
For each integer i let U_i be the collection of size i subsets $\{t_1, ...t_i\} \subset 2^{<\omega}$ such that for any $b_1 \sqsupset t_1, ...b_i \sqsupset t_i$, $\{b_l, b_m\}$ is not an edge for $0 < l \neq m \leq i$.
For each tuple $\tau = \{t_1, ...t_i\} \in U_i$, let Q_τ be the collection of all size i subsets $\{b_1, ...b_i\} \subset n^\omega$ such that $b_1 \sqsupset t_1, ..., b_i \sqsupset t_i$. $\square$

Thank you!