

Day Five

Stability and Independence

Using dividing, define A is independent from B over C , and we write $A \downarrow_C B$

Theorem E.1 Let $\mathcal{U}$ be a K -universal domain for T . If T is stable, then the independence relation $\downarrow$ defined using nondividing satisfies the following 7 pptides.

- (1) Invariance under $\text{Aut}(\mathcal{U})$
- (2) Symmetry: $A \downarrow_C B \Leftrightarrow B \downarrow_C A$
- (3) Transitivity
- (4) Finite character
- (5) Extension

- (6) local character
(7) stationarity of types.

Pf: ~~by~~

□

A relation satisfying (4) - (7) is called a stable independence relation on $\mathcal{U}$.

Theorem E.2 Let $\mathcal{U}$ be a $\mathcal{K}$ -universal domain for T . If T is stable, there is precisely one stable independence relation on $\mathcal{U}$. Moreover, if there exists a stable independence relation $A \downarrow_c^* B$ on triples of small subsets of $\mathcal{U}$, then T is stable. □

Examples:

- 1) orthogonality in Hilbert spaces
- 2) probabilistic independence in probability spaces.

Model theory of probability algebras

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

For all $a, b \in \mathcal{F}$, define $a \equiv_{\mu} b$ if $\mu(a \Delta b) = 0$.

Then $\equiv_{\mu}$ is equivalence relation. We denote the equivalence class of a under $\equiv_{\mu}$ by $[a]_{\mu}$. Let $\hat{\mathcal{F}}$ denote the set $\{[a]_{\mu} \mid a \in \mathcal{F}\}$.

We call $(\hat{\mathcal{F}}, \mu)$ the probability algebra associated to $(\Omega, \mathcal{F}, \mu)$.

We define $d(a, b) := \mu(a \Delta b)$ for all $a, b \in \hat{\mathcal{F}}$.

Fact: $(\hat{\mathcal{F}}, d)$ is a complete metric space.

$L_{pr} = \{0, 1, \cdot^c, \wedge, \vee, \mu\}$, where 0

and 1 are constant symbols, $\cdot^c$ is a unary function symbol, $\cap$ and $\cup$ are binary function symbols, and μ is a unary predicate symbol. $\cdot^c$ and μ are 1 -Lipschitz, $\cap$ and $\cup$ are 2 -Lipschitz.

The theory P_r consists of the following

axioms:

(i) Boolean algebra axioms.

$$\sup_x \sup_y (d(x \cup y, y \cup x)) = 0 \leq \forall x \forall y x \cup y = y \cup x$$

$$\sup_x \sup_y (d(x \cap y, y \cap x)) = 0$$

$$\sup_x \sup_y \sup_z d(x \cup (y \cup z), (x \cup y) \cup z) = 0$$

$$\sup_x \sup_y d(x \cup (x \cap y), x) = 0; \sup_x \sup_y d(x \cap (x \cup y), x) = 0$$

$$\sup_x \sup_y \sup_z d(x \cup (y \cap z), (x \cup y) \cap (x \cup z)) = 0$$

$$\sup_x \sup_y \sup_z d(x \cap (y \cup z), (x \cap y) \cup (x \cap z)) = 0$$

$$\sup_x d(x \cup x^c, 1) = 0; \quad \sup_x d(x \cap x^c, 0) = 0$$

(ii) Measure axioms

$$\mu(0) = 0 \text{ and } \mu(1) = 1$$

$$\sup_x \sup_y \left| \frac{\mu(x \cup y) + \mu(x \cap y)}{2} - \frac{\mu(x) + \mu(y)}{2} \right| = 0$$

(iii) μ and d .

$$\sup_x \sup_y |d(x, y) - \mu(x \Delta y)| = 0.$$

We interpret $\hat{\mathcal{F}}$ as an L_{Pr} -structure.

$\mathcal{M} = (\hat{\mathcal{F}}, 0, 1, \cdot^c, \cup, \cap, \mu, d)$ naturally.

We call such $\mathcal{M}$ a probability structure.

Theorem E.3 Let $\mathcal{M}$ be an L_{Pr} -structure.

Then $\mathcal{M} \models Pr$ iff $\mathcal{M}$ is isomorphic to the probability structure associated to a probability space $(\Omega, \mathcal{F}, \mu)$.

Note: Pr is not complete.

$$(iv) \sup_x \inf_y |\mu(x \cap y) - \mu(x \cap y^c)| = 0$$

Let $APr = Pr \cup \{(iv)\}$.

Corollary E.4 The model of APr are

isomorphic to the probability structures associated to atomless probability spaces.

Theorem E.5 (Ben Yaacov) Let $M \models \text{APr}$.
Then (i) APr is separably categorical and complete.
(ii) APr admits quantifier elimination.

(iii) The universal part of APr is Pr and
 APr is the model completion of Pr .

(iv) Assume that M is the probability structure associated to a probability space $(\Omega, \mathcal{F}, \mu)$.
Let $\mathcal{C} \subseteq \mathcal{M}$ and let $\mathcal{G}$ be the σ -algebra σ -generated by the events in $\mathcal{C}$. For all $a, b \in M^n$,
 $\text{tp}(a/\mathcal{C}) = \text{tp}(b/\mathcal{C})$

iff $\mathbb{P}(a_1^{\varepsilon_1} \wedge \dots \wedge a_n^{\varepsilon_n} | \mathcal{G}) = \mathbb{P}(b_1^{\varepsilon_1} \wedge \dots \wedge b_n^{\varepsilon_n} | \mathcal{G})_{\text{a.s.}}$
for all $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$

(v) APr is ω -stable and its independence relation coincides with the probability independence relation.

That is, if A, B, C are subsets of a model of $\mathcal{A}pr$, then

$$A \perp_c B \Leftrightarrow \mathbb{P}(a \cap b | \sigma(C)) = \mathbb{P}(a | \sigma(C)) \cdot \mathbb{P}(b | a) \quad \text{a.s.}$$

for all $a \in \sigma(A)$ and $b \in \sigma(B)$. $\square$

Prop: $A \perp_c B \Leftrightarrow \mathbb{P}(a | \sigma(B) \vee \sigma(C)) = \mathbb{P}(a | \sigma(C))$
a.s.
for all $a \in \sigma(A)$.

Universality, homogeneity and saturation

Defn (Hoover - Keisler) Let X be a

Polish metric space. A probability space Ω is called X -universal if for all X -valued random variables y on an arbitrary prob. space $\mathbb{P}$, there is an X -valued random

variable x on Ω , such that x and y have the same distribution. We call Ω universal if it is X -universal for every Polish metric space X .

Theorem E.6 Let Ω be a probability space. TFAE.

- (i) Ω is universal.
- (ii) Ω is $[0,1]$ -universal.
- (iii) Ω is $[0,1]^{\mathbb{N}}$ -universal.
- (iv) Ω is atomless.

Prop E.7. (S.) Let $\mathcal{A} = (A, \mu, d)$ be an atomless probability structure.

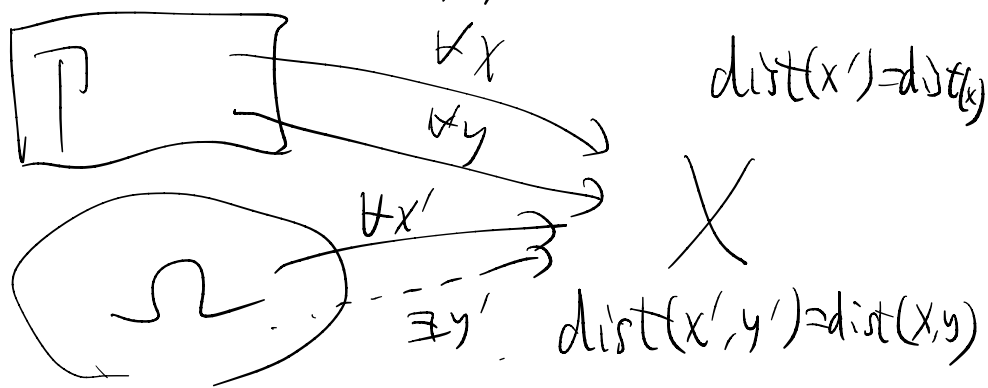
TFAE.

- (i) A is τ -homogeneous as a complete boolean algebra.
- (ii) A is homogeneous as a complete boolean algebra.
- (iii) A is strongly $\aleph_1$ -homogeneous as an L_{pr} -structure
- (iv) A is strongly ω -homogeneous as an L_{pr} -structure. □

A probability space Ω is κ -saturated if its probability algebra is κ -saturated as an L_{pr} -structure.

Defn (Hoover-Keisler) Let Ω and Γ be probability spaces and let X be a Polish

metric space. Let $x, y: \mathcal{P} \rightarrow X$ be random variables. The space Ω is said to have the saturation ppty for $\text{dist}(x, y)$ if for every random variable $x': \Omega \rightarrow X$ with $\text{dist}(x') = \text{dist}(x)$, there is a random variable $y': \Omega \rightarrow X$ s.t. $\text{dist}(x, y) = \text{dist}(x', y')$.



A probability space Ω is said to be Hoover-Kiesler saturated if for all two random variables $x, y: \mathcal{P} \rightarrow X$, where $\mathcal{P}$ is an arbitrary prob. space, and X is an arbitrary Polish metric space,

Ω has the saturation ppty of $\text{dist}(x, y)$.

Defn: A measure space $(X, \mathcal{A}, \mu)$ is atomless if for every $a \in \mathcal{A}$ with $\mu(a) > 0$, there exists $b \in \mathcal{A}$ s.t. $b \subseteq a$ and $0 < \mu(b) < \mu(a)$.

If $\mathcal{B} \subseteq \mathcal{A}$ is a σ -subalgebra, we say that

$\mathcal{A}$ is atomless over $\mathcal{B}$, if for every $a \in \mathcal{A}$ with $\mu(a) > 0$, there is $b \in \mathcal{A}$, such that $\forall c \in \mathcal{B}$, $a \cap b \neq a \cap c$.

$\mathcal{A}$ is q -atomless over $\mathcal{B}$, if $\mathcal{A}$ is atomless over $\sigma(\mathcal{B} \cup \mathcal{S})$ for all $|\mathcal{S}| < q$.

Theorem E.8 (Maharam) For all atomless probability space Ω , there is a countable set of distinct infinite cardinals $S = \{\aleph_i \mid i \in I\}$ such that the measure algebra of Ω is isomorphic to a convex combination of the homogeneous prob space $[0,1]^{\aleph_i}$. The set S is unique determined by Ω and is called the Maharam spectrum.

$$\frac{1}{2} \circlearrowleft [0,1] + \frac{1}{4} \circlearrowleft [0,1]^{\aleph_3} + \frac{1}{4} \circlearrowleft [0,1]^{\aleph_5}$$

homogeneous
prob space

Theorem E.9 (S.) Let Ω be an atomless probability space and let κ be an infinite cardinal. Then TFAE

- (i) Ω is κ -saturated
- (i') Every cardinal in the Maharam spectrum of Ω is $\geq \kappa$.
- (ii) Ω is κ -atomless.

Corollary E.10 (S.) An atomless probability space is Hoover-Keisler saturated iff it is $\aleph_1$ -saturated.

Theorem E.11 (S.) For all infinite cardinals κ , the probability algebra of $[0, 1]^\kappa$ is strongly κ -homogeneous and κ -saturated. Moreover, it is the unique model of APr of density character κ with these properties.

Model theory of random variable structures

A $([0,1])$ -valued random variable structure is based on the set of the form $M = L^1(\Omega, \mathcal{F}, \mu, [0,1])$, where $(\Omega, \mathcal{F}, \mu)$ is a probability space. It is called atomless random variable structure, if its underlying probability space is atomless.

$L_{RV} = \{0, \gamma, -, \frac{1}{2}, I\}$, where 0 is a constant symbol, $-$ is a binary function symbol, γ and $\frac{1}{2}$ are unary function symbols, and I is a unary predicate symbol. γ is 1-Lipschitz, $\frac{1}{2}$ is $\frac{1}{2}$ -Lipschitz, $-$ is 2-Lipschitz, and I is 1-Lipschitz.

We interpret the symbols of L_{PV} in M as follows.

$$1) 0^M(\omega) = 0 \text{ for all } \omega \in \Omega$$

$$2) 1^M(f) = 1 - f \text{ for all } f \in M.$$

$$3) (-)^M(f, g) = f \dot{-} g = \max\{f - g, 0\} \text{ for all } f, g \in M$$

$$4) (\frac{1}{2})^M(f) = \frac{f}{2} \text{ for all } f \in M.$$

$$5) I^M(f) = \int_{\Omega} f d\mu \text{ for all } f \in M.$$

$$6) d^M(f, g) = \int_{\Omega} |f - g| d\mu \text{ for all } f, g \in M.$$

Then $M = (L'(\Omega, \mathcal{F}, \mu), [0, 1], 0, 1, \dot{-}, \frac{1}{2}, I, d)$ is an L_{PV} -structure and is denoted by

$$L'(\Omega, \mathcal{F}, \mu, [0, 1]).$$

Notation:
$$\begin{aligned} X \wedge Y &= X \dot{-} (X \dot{-} Y) \\ X \vee Y &= 1 \dot{-} (1 \dot{-} X \wedge 1 \dot{-} Y) \end{aligned}$$

Ben Yaacov's axiomatizations

$$(RV1) \sup_x \sup_y \frac{1}{2} | I(x) - (I(x \dot{-} y) + I(y \wedge x)) | = 0$$

$$(RV2) | I(1) - 1 | = 0$$

$$(RV3) \sup_x \sup_y \frac{1}{2} | d(x, y) - (I(x \dot{-} y) + I(y \dot{-} x)) | = 0$$

$$(RV4.1) \sup_x \sup_y d((x \dot{-} y) \dot{-} x, 0) = 0$$

$$(RV4.2) \sup_x \sup_y \sup_z d((x \dot{-} z) \dot{-} (x \dot{-} y) \dot{-} (y \dot{-} z), 0) = 0$$

$$(RV4.3) \sup_x \sup_y d((x \wedge y) \dot{-} (y \wedge x), 0) = 0$$

$$(RV4.4) \sup_x \sup_y d((x \dot{-} y) \dot{-} (y \dot{-} x), 0) = 0$$

$$(RV4.5) \sup_x d\left(\left(\frac{x}{2} \dot{-} \left(x \dot{-} \frac{x}{2}\right)\right), 0\right) = 0$$

$$(RV4.6) \sup_x d\left(\left(x \dot{-} \frac{x}{2}\right) \dot{-} \frac{x}{2}, 0\right) = 0$$

Some come from Chang's MV-algebra.

Let RV be the theory consists of the above axioms.

$$(NA) \sup_x \inf_y \max(I(y \wedge \neg y), |I(y \wedge x) - \frac{I(x)}{2}|) = 0$$

$$ARV = RV \cup \{(NA)\}$$

Prop E.12 (Ben Yaacov) Let $\mathcal{M}$ be an L_{RV} -structure. Then $\mathcal{M}$ is a model of RV iff $\mathcal{M}$ is isomorphic to $(L^1(\Omega, \mathcal{F}, \mu), [0, 1], \wedge, \vee, \neg, I, d)$ for some probability space $(\Omega, \mathcal{F}, \mu)$. Moreover, $\mathcal{M}$ is a model of ARV iff $(\Omega, \mathcal{F}, \mu)$ is an atomless probability space.

pf: Use Chang's completeness theorem. $\square$
≥ 50 pages

S. gave an elementary approach to axiomatizing the class of random variable structures, but the axioms are longer.

Theorem E.13 Every L_{Pr} -isomorphism

$\varphi: A \rightarrow A'$ induces an L_{RV} -isomorphism

$$R(\varphi): L'((\Omega, \mathcal{F}, \mu), (\omega, 1)) \rightarrow L'((\Omega', \mathcal{F}', \mu'), (\omega, 1))$$

On the other hand, every L_{RV} -isomorphism

$$\psi: L'((\Omega, \mathcal{F}, \mu), (\omega, 1)) \rightarrow L'((\Omega', \mathcal{F}', \mu'), (\omega, 1))$$

induces an L_{Pr} -isomorphism $P(\psi): A \rightarrow A'$.

Moreover, $R(P(\psi)) = \psi$ and $P(R(\varphi)) = \varphi$.

Rnk : RV and Pr are biinterpretable.

Theorem E.14 (Ben Yaacov)

Let $M = L'(\mathcal{L}, \mathcal{F}, \mu, \omega, \nu) \models \text{ARV}$.

(i) ARV is separably categorical and complete

(i') ARV admits quantifier elimination

(i'') The universal part of ARV is RV and

ARV is the model completion of RV.

(iv) Two tuples f and g in M^n have the same type over a set $A \subseteq M$ iff they have the same joint conditional distribution over $\sigma(A)$.

That is, $\text{tp}(f/A) = \text{tp}(g/A)$

$\Leftrightarrow \text{dist}(f | \sigma(A)) = \text{dist}(g | \sigma(A))$

(v) ARV is ω -stable, and the independence

coincides with probabilistic independence.
i.e., $A \overset{\text{ARV}}{\perp\!\!\!\perp}_B C$ iff $\sigma(A) \overset{\text{Apr}}{\perp\!\!\!\perp}_{\sigma(B)} \sigma(C)$.

Defn: Let $\mathcal{A}$ be a probability algebra.

An n -dimensional conditional distribution over

$\mathcal{A}$, μ , is an $L^1(\mathcal{A}, \mathbb{C}, 1)$ -valued Borel probability measure on $\mathbb{R}^n$. More precisely,

1) $\mu(B) \geq 0$ a.s. for all Borel sets $B \subseteq \mathbb{R}^n$.

2) $\mu(\mathbb{R}^n) = 1$ a.s.

3) $\mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i)$ a.s. for all disjoint

Borel sets $B_1, B_2, \dots \subseteq \mathbb{R}^n$.

Let $\mathcal{D}_{\mathbb{R}^n}(\mathcal{A})$ denote the space of all

n -dimensional conditional distributions over A .
For every Borel set $B \subseteq \mathbb{R}^n$, we denote by $\mathcal{D}_B(A)$ the space of all n -dimensional conditional distributions over A which are supported by B , as measures.

We usually let $B = [0, 1]^n$.

Theorem E.15 [BBH] The mapping

$$\begin{aligned} \gamma_n : S_n(A) &\rightarrow \mathcal{D}_{[0,1]^n}(A) \\ t_p(f/A) &\mapsto \text{dist}(f|A) \end{aligned}$$

is a homeomorphism between $S_n(A)$ equipped with the logic topology and $\mathcal{D}_{[0,1]^n}(A)$ equipped with the topology of weak convergence.

For all $p, q \in S_n(T_A)$, we define $d^*(p, q)$ to be
 $\inf \left\{ \sum_{i=1}^n d(b_i, c_i) \mid \mu_A \models p[b_1, \dots, b_n] \text{ and } \mu_A \models q[c_1, \dots, c_n] \right\}$

Note: $d(a, b) \leq d^*(a, b) \leq nd(a, b)$.
 equivalent metrics.

Defn: $\mathcal{P}_1([0, 1]^n) = \{ \text{probability measures on } [0, 1]^n \}$
 $\mathcal{D}([0, 1]^n)$

$$W_1(\mu, \nu) = \inf \left\{ \mathbb{E} d(X, Y) \mid \begin{array}{l} \text{dist}(X) = \mu \\ \text{dist}(Y) = \nu \end{array} \right\}$$

Wasserstein distance.

$(\mathcal{P}_1([0, 1]^n), W_1)$ is called Wasserstein space.

Optimal transport.

Theorem E.16 (S.) The mapping

$$\eta_n : S_n(\text{ARV}) \rightarrow P_1([0, 1]^n)$$

$$tp(f) \mapsto \text{dist}(f)$$

is an isometric isomorphism between $(S_n(\text{ARV}), d^*)$

and $(P_1([0, 1]^n), W_1)$. □

Consider $S_n(\text{ARV}_C)$. We may define conditional Wasserstein distances.

Theorem E.17 (weak version) (S.)

Let $M \models \text{ARV}$ be a κ -universal domain.

Suppose $C \subseteq M$ is small and let $\mathcal{C}$ be the σ -algebra of measurable sets generated by the random variables on C .

Let $f, g \in M$, Then

$$d(tp(f/c), tp(g/c))$$

$$= \int_0^1 \|P(f > t | \mathcal{L}) - P(g > t | \mathcal{L})\|_1 dt.$$

This generalizes the following result.

Theorem E.18 For all $\mu, \nu \in P(\mathbb{R})$, one has

$$W_1(\mu, \nu) = \int_{-\infty}^{\infty} |F(x) - G(x)| dx,$$

where F and G are distributions for μ and ν respectively.

Using Kantorovich-Rubinstein duality formula, we may generalize Thm E.18 to the most general case, i.e., $d^*(p, q)$, where $p, q \in S_n(ARV_c)$.

Theorem E.19 (S.) No separable model of ARV is $\aleph_0$ -saturated.

Theorem E.20 (S.) Let $M \models \text{ARV}$ be of the form $L'(\mathcal{M}, [\mathcal{O}, 1])$, where $(\Omega, \mathcal{F}, \mu)$ is an atomless probability space. For every uncountable cardinal κ ,

M is κ -saturated iff

$\hat{\mathcal{F}}$ is a κ -saturated probability algebra.

Theorem E.21 (S.) Let $M \models \text{ARV}$ be of the form $L'(\mathcal{M}, [\mathcal{O}, 1])$, where $(\Omega, \mathcal{F}, \mu)$ is an atomless probability space. Then TFAE:

- (i) Ω is Hoover-Keisler saturated
- (i') Ω is $\aleph_1$ -saturated.

(iii) Every cardinal in the Maharam spectrum of Ω is uncountable

(iv) $\mathcal{M}$ is $\mathfrak{h}_1$ -saturated

(v) $\mathcal{M}$ is $\mathfrak{h}_0$ -saturated

(vi) For all $a, b, c \in \mathcal{M}$ with $tp(a) = tp(b)$, there exists $d \in \mathcal{M}$ such that $tp(a, c) = tp(b, c)$
"2-saturated"

Theorem E.22(S.) For every uncountable cardinal κ , we have that $L'([0, 1]^{\kappa}, A_{\kappa}, \mu_{\kappa}, \mathbb{Q}, 1) \models ARV$ is κ -saturated and strongly κ -homogeneous. Moreover, it is the unique model of ARV of density character κ with these properties.

On d-finite tuples $\leadsto$ finite tuples
in FOL.

Defn: Let a and c be tuples and let
 $p = tp(a/c)$. Then a is called d-finite over
 c , equivalently, the type p is called
d-finite, if for every tuple b and
every $\varepsilon > 0$, there is $\delta = \delta_{b, \varepsilon}^{a, c} > 0$ such that
whenever $a' \equiv_c a$, and $d(a, a') \leq \delta$ there
is b' such that $d(b, b') \leq \varepsilon$ and
 $a'b' \equiv_c ab$. (i.e., $tp(a'b'/c) = tp(ab/c)$).
If $c = \emptyset$, we omit c from the notation.

Thm E.23 (B.U.)

Let T be a separably categorical L -theory for a countable L . Let $M \models T$ and let f be a finite tuple in M . We let $T(f)$ denote the theory $\text{Th}(M, f)$. Then f is d -finite iff $T(f)$ is separably categorical.

Fact: Every finite tuple in a model of APr is d -finite.

Theorem E.24 (S.) Let $M \models \text{ARV}$ and $f = (f_1, \dots, f_n) \in M^n$. Then f is d -finite iff f_i is a discrete random variable for each i .

Prop E.25 (S.) let $M \models ARV$ and let f be a tuple in M . Then

- (i) If f is d-finite, then $ARV(f)$ has a unique separable model up to isomorphism
- (ii) If f is not d-finite, then $ARV(f)$ has continuum many non-isomorphic separable models.

This work has applications to Keisler and Sun's work.

Simple $\longrightarrow$ Discrete.
Generalized

To add:

1) proof systems.

e.g. $MIP^* = RE$ (Goldbring-Hart)

2) Completeness Theorem in CFO.