Day Four

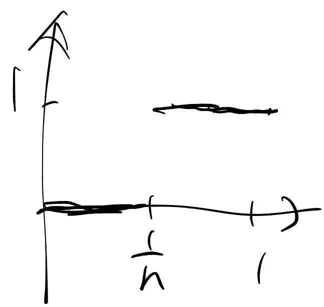
Some answers:

(1) Why we consider uniform cont functions f with the same Δ_f.

(a) What if $\{f_n\}$ are not continuous

$f_n : [0,1] \to [0,1]$

$$f_n(x) = \begin{cases} 0, & 0 \le x < \frac{1}{n} \\ 1, & \frac{1}{n} \le x \le 1. \end{cases}$$



D is an ultrafilter on $\mathbb{N}$.

$$\prod_D f_n : \prod_D [0,1] \to [0,1]$$

$(0, 0, \cdots, 0, \cdots) \sim (0, 1, \frac{1}{2}, \cdots, \frac{1}{n} \cdots)$

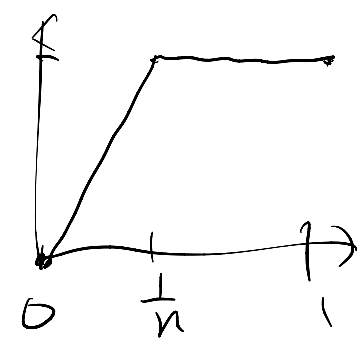
$\prod_b f_n((0, 0, \cdots, 0, \cdots)) = 0$

$\prod_b f_n((0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots)) = 1$

So $\prod_b f_n$ is not well-defined.

(b) Even $f_n : [0,1] \to [0,1]$ are unif. continuous but Δf_n are different.

$f_n(x)$'s graph.



Still

$\prod_b f_n((0, \cdots, 0, \cdots)) = 0$

$\prod_b f_n((0, \cdots, \frac{1}{n}, \cdots)) = 1$.

(2) Consider the theory of L^p Banach lattices

$T' = Th(L^p[0,1], \chi_{[0,1]})$, a new constant.

T' has exactly two separable models

$(L^p[0,1], \chi_{[0,1]})$; $(L^p[0,2], \chi_{[0,1]})$.

$tp(\chi_{[1,2]} / \chi_{[0,1]})$ is realized or not.

Vaught's Thm fails in CFO.

However, Ben Yaacov and Usuyatsov,

On d-finiteness (?) Fund. Math.

<u>Thm</u>: Assume T has enought d-finite elements.

Then T cannot have precisely two non-isomorphic separable models.

(3) FOL $\mathcal{M}$ $\rightarrow$ considered as a metric structure.

$\sup = \forall$

$\inf = \exists$.

Functions on type spaces

Let $\mathcal{M}_A = (\mathcal{M}, a)_{a \in A}$ be a model of T_A in which each type in $S_n(T_A)$ is realized, for each $n \geq 1$. Let $\varphi(x_1, \dots, x_n)$ be an $L(A)$-fla.

For each type $p \in S_n(T_A)$, $\tilde{\varphi}(p) = r$, where r is the unique real number s.t. $\varphi = r \in p$.

Equivalently, $\tilde{\varphi}(p) = \varphi^{\mathcal{M}}(b)$ where $b \models p$.

<u>Lemma D.1</u>: Let $\varphi(x_1, \dots, x_n)$ be an $L(A)$-fla.

Then $\tilde{\varphi} : S_n(T_A) \to [0,1]$ is continuous for the logic topology and uniformly continuous for the d-metric distance on $S_n(T_A)$.

Pf: (1) $\tilde{\varphi}^{-1}(r-\varepsilon, r+\varepsilon) = [\,|\varphi \dot{-} r| < \varepsilon\,]$.

(2) Δ_φ for $\varphi^{\mathcal{M}}$ is also a modulus of u.c. for $\tilde{\varphi}$. $\square$

<u>Prop D.2</u> Let $\Phi : S_n(T_A) \to [0,1]$. TFAE

(1) Φ is continuous for the logic topology on $S_n(T_A)$.

(2) There is a sequence $(\varphi_k(x_1, \cdots, x_n) \mid k \geq 1)$ of $L(A)$-flas such that $(\varphi_k \mid k \geq 1) \rightrightarrows \Phi$ on $S_n(T_A)$

(3) Φ is continuous for the logic topology and uniformly continuous for the d-metric on $S_n(T_A)$.

Pf: (1) $\Rightarrow$ (2) Lattice version of the Stone-Weierstrass Thm.

(2) $\Rightarrow$ (3) φ_k is cont. in logic topology and unif. cont. in d-metric. Preserved under uniform convergence.

(3) $\Rightarrow$ (1) trivial. $\square$

Definability in metric structures

definable predicates $\rightsquigarrow$ definable sets

$\rightsquigarrow$ definable functions

Definable predicates

A predicate $P: M^n \to [0,1]$ is <u>definable</u> in $\mathcal{M}$ over A if there is a seq $(\varphi_k(x) \mid k \geq 1)$ of $L(A)$-flas. s.t. $\varphi_k^{\mathcal{M}}(x) \rightrightarrows P(x)$ on M^n.

i.e., $\forall \varepsilon > 0\ \exists N\ \forall k \geq N\ \forall x \in M^n\ |\varphi_k^{\mathcal{M}}(x) - P(x)| \leq \varepsilon$.

This might show that why we say that connective are too restricted.

We enlarge our connectives to include all continuous $u: [0,1]^{\mathbb{N}} \to [0,1]$.

Define $\rho: [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} \to [0,1]$

$$((a_k), (b_k)) \mapsto \sum_{k=0}^{\infty} 2^{-k} |a_k - b_k|$$

$([0,1]^{\mathbb{N}}, \rho)$ is a compact metric space.

<u>Prop D.3</u> Let $\mathcal{M}$ be an L-structure with $A \subseteq M$ and suppose $P: M^n \to [0,1]$ is a predicate. Then

P is definable in $\mathcal{M}$ over A

iff there is a continuous function $u: [0,1]^{\mathbb{N}} \to [0,1]$ and $L(A)$-fla seq $(\varphi_k \mid k \geq 1)$ s.t. $\forall x \in M^n$

$$P(x) = u(\varphi_k^{\mathcal{M}}(x) \mid k \in \mathbb{N})$$

Pf: Uses Tietze Extension Thm $\square$

Rmk: a is completely independent of p.

check: forced limit of flas in [BU].

Lemma D.4 Suppose $p: M^n \to [0,1]$ is definable in $\mathcal{M}$ over A and consider $\mathcal{N} \preceq \mathcal{M}$ with $A \subseteq N$. Then $\inf_x p(x)$ and $\sup_x p(x)$ have the same value in $\mathcal{N}$ as in $\mathcal{M}$.

Pf: Easy. $\square$

Prop D.5 Let $p_i : M^n \to [0,1]$ be definable in $\mathcal{M}$ over A for $i = 1, \dots, m$ and consider

$\mathcal{N} \preceq \mathcal{M}$ with $A \subseteq N$. Let Q_i be $P_i|_{N^n}$ for each i. Then $(\mathcal{N}, Q_1, \dots, Q_m) \preceq (\mathcal{M}, P_1, \dots, P_m)$.

Pf: 略 $\square$

<u>Prop D.6</u>: Let $P: M^n \to [0,1]$ be definable in $\mathcal{M}$ over A and consider an elementary extension $\mathcal{N}$ of $\mathcal{M}$. There is a unique predicate $Q: N^n \to [0,1]$ such that Q is definable in $\mathcal{N}$ over A and P is the restriction of Q to M^n. This predicate satisfies

$$(\mathcal{M}, P) \preceq (\mathcal{N}, Q).$$

Pf: 略. $\square$

Thm D.7 Let $P : M^n \to [0,1]$ be a function. Then P is a predicate definable in $\mathcal{M}$ over A iff there is $\Phi : S_n(T_A) \to [0,1]$ that is continuous w.r.t. the logic topology on $S_n(T_A)$ s.t. $P(a) = \Phi(tp_{\mathcal{M}}(a/A))$ for all $a \in M^n$.

$$\begin{tikzcd} M^n \arrow[d, "tp(\cdot/A)"'] \arrow[dr, "\forall \mathrm{def}\ P"] & \\ S_n(T_A) \arrow[r, dashed, "\exists \Phi"'] & {[0,1]} \end{tikzcd}$$

Pf: 略. $\square$

A subset $S \subseteq M^n$ is <u>type-definable</u> in $\mathcal{M}$ over A if there is a set $\Sigma(x_1, \ldots, x_n)$ of $L(A)$-flas such that for every $a \in M^n$ we have $a \in S$ iff $\varphi^{\mathcal{M}}(a) = 0$ for ever $\varphi \in \Sigma$. We say that S is type-defined by Σ.

In saturated models, definable predicates have a new characterisation.

<u>Corollary D.8</u> Let $\mathcal{M}$ be a κ-saturated L-structure and $A \subseteq M$ with $\operatorname{card}(A) < \kappa$; let $P : M^n \to [0,1]$ be a function. Then, we have:

P is a predicate definable in $\mathcal{M}$ over A iff the sets $\{a \in M^n \mid P(a) \le r\}$ and $\{a \in M^n \mid P(a) \ge r\}$ are type-definable in $\mathcal{M}$ over A for every $r \in [0,1]$.

Pf: $\cancel{PD}_k$. $\square$

Distance predicates

Let $D \subseteq M^n$. Define $\mathrm{dist}(x, D) = \inf\{d(x,y) \mid y \in D\}$.

It's a map: $M^n \to [0,1]$.

$\forall\, x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in M^n$

$$d(x,y) = \max(d(x_1,y_1), \dots, d(x_n,y_n)).$$

$\mathrm{dist}(x, D)$ can be characterized by axioms in CFO.

Consider a predicate $p : M^n \to [0,1]$

(E1) $\sup_x \inf_y \max\big(p(y), |p(x) - d(x,y)|\big) = 0$

(E2) $\sup_x \big| p(x) - \inf_y \min\big(p(y) + d(x,y), 1\big)\big| = 0$

<u>Observation:</u> $\forall D \overset{\neq \emptyset}{\subseteq} M^n$, $p(x) = \mathrm{dist}(x, D)$ satisfies E1 and E2.

<u>Thm D.9</u> Let (M, F) be an L-structure satisfying E1 and E2. Let

$D = \{x \in M^n \mid F(x) = 0\}$ be the zeroset of F.

Then $F(x) = \mathrm{dist}(x, D)$ for all $x \in M^n$.

Pf: 略. $\square$

Zerosets $\approx$ type-definable sets $\subseteq$ def sets

Defn: Let $D \subseteq M^n$. We say that D is a **zeroset** in $\mathcal{M}$ over A if there is a predicate $P : M^n \to [0,1]$ definable in $\mathcal{M}$ over A s.t. $D = \{x \in M^n \mid P(x) = 0\}$.

Prop D.10 TFAE

(1) D is a zeroset in $\mathcal{M}$

(2) there is a sequence $(\varphi_m \mid m \geq 1)$ or L-flas. s.t. $D = \{x \in M^n \mid \varphi_m^{\mathcal{M}} = 0 \quad \forall m \in \mathbb{N}\}$

$$= \bigcap_{m=1}^{\infty} \text{zero set of } \varphi_m^{\mathcal{M}}.$$

Pf: (1) $\Rightarrow$ (2) Take L-flas seq $(\varphi_m \mid m \geq 1)$ s.t. $\forall x \in M^n$, $\forall m$ $|P(x) - \varphi_m^{\mathcal{M}}(x)| \leq \frac{1}{m}$.

Then $D = \bigcap_m D_m$, where D_m is the zeroset of $(\varphi_m(x) \dot{-} \frac{1}{m})$ in $\mathcal{M}$.

(2) $\Rightarrow$ (1) $P(x) = \sum_{m=1}^{\infty} 2^{-m} \varphi_m^{\mathcal{M}}(x)$ is a def predicate. Then, D is zeroset of P $\square$

<u>Corollary D.11</u>: The collection of zerosets in $\mathcal{M}$ over A is closed under countable intersections. $\square$

<u>Definable sets</u>

A closed set $D \subseteq M^n$ is <u>definable</u> in $\mathcal{M}$

over A if the distance predicate $\mathrm{dist}(x, D)$ is definable in $\mathcal{M}$ over A.

<u>Thm D.12</u>. Let $D \subseteq M^n$ be a closed set. TFAE

(1) D is definable in $\mathcal{M}$ over A

(2) For every predicate $P: M^m \times M^n \to [0,1]$ that is definable in $\mathcal{M}$ over A, the predicate $Q: M^m \to [0,1]$ defined by $Q(x) = \inf\{P(x,y) \mid y \in D\}$ is definable in $\mathcal{M}$ over A.

Pf: See Thm 9.17 in [BBHU]. $\square$

<u>Prop D.13</u> Let $\mathcal{N} \preceq \mathcal{M}$ be L-structs

and Let $D \subseteq M^n$ be definable in $\mathcal{M}$ over A, where $A \subseteq N$. Then

(1) $\forall x \in N^n$, $\mathrm{dist}(x, D) = \mathrm{dist}(x, D \cap N^n)$.
Thus, $D \cap N^n$ is definable in $\mathcal{N}$ over A.

(2) $(\mathcal{N}, \mathrm{dist}(\cdot, D \cap N^n)) \preceq (\mathcal{M}, \mathrm{dist}(\cdot, D))$

(3) If $D \neq \emptyset$, then $D \cap N^n \neq \emptyset$.

Pf: Prop 9.18 in [BBHU]. $\square$

Def sets $\Rightarrow$ zero sets.

Prop D.14 Let $D \subseteq M^n$ be a closed set. TFAE.

(1) D is definable in $\mathcal{M}$ over A.

(2) There is a predicate $P : M^n \to [0,1]$ definable in $\mathcal{M}$ over A such that $P(x) = 0$ for all $x \in D$.
($D \subseteq$ zero set of P)

and $\forall \varepsilon>0 \exists \delta>0 \forall x \in M^n \, [\, P(x) \leq \delta \Rightarrow dist(x, D) \leq \varepsilon \,]$

(3) There is a seq $(\varphi_m \mid m \geq 1)$ of $L(A)$-flas and a seq $(\delta_m \mid m \geq 1)$ of positive real numbers such that for all $m \geq 1$ and $x \in M^n$

$(x \in D \Rightarrow \varphi_m^M(x) = 0)$ ($D \subseteq$ zero set of φ_m^M)

and $(\varphi_m^M(x) \leq \delta_m \Rightarrow dist(x, D) \leq \frac{1}{m})$.

Pf: prop 4.19 in [BBHU]. $\blacksquare$

Related to Gromov–Hausdorff metric.

In ω_1-saturated L-structures.

<u>Prop D.15</u>: Let M be an ω_1-saturated L-structure and let $P: M^n \to [0,1]$ be a predicate definable in M over A. Then

The zeroset D of P is definable in $\mathcal{M}$ over A

$\Leftrightarrow \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in M^n \, (p(x) \leq \delta \Rightarrow \mathrm{dist}(x, D) \leq \varepsilon)$

pf: 9.20 in [BBHU]. $\square$

<u>Ex</u>: $B(a,r) = \{x \in M^n \mid d(a,x) \leq r\}$

$B(a,r)$ is definable in $\mathcal{M}$ over $\{a\}$

$\Leftrightarrow B(a, r+\delta) \to B(a,r)$ in the sense of Gromov-Hausdorff metric, as $\delta \to 0$.

(let $p(x) = d(a,x) \dot{-} r$, then $p(x) \leq \delta$
$\Leftrightarrow d(a,x) \leq r+\delta$)

In the case of compact subset, we have:

Prop D.16: Let $\mathcal{M}$ be an ω_1-saturated L-structure and $A \subseteq M$. Let $C \subseteq M^n$ be compact. TFAE.

(1) C is a zeroset in $\mathcal{M}$ over A.

(2) C is definable in $\mathcal{M}$ over A.

Pf: Prop 10.6 in [BBHU]. $\square$

Also, in ω-categorical theory, definable sets have some better characterization,

Definability of FOL and CFO

Let $\mathcal{M}$ be an FOL structure.

Define $d(x,y)=1$ if $x\neq y$
$d(x,y)=0$ if $x=0$.

Then $\mathcal{M}$ is also a metric space.

And we can view $\mathcal{M}$ as a CFO struct.

There are more CFO flas than FOL flas.

<u>Fact</u> 1) for every L-fla φ, $\{\varphi^{\mathcal{M}}(x) \mid x \in M^n\}$ is a finite set.

2) for every $r \in [0,1]$, $\{x \in M^n \mid \varphi^{\mathcal{M}}(x) = r\}$ is definable in $\mathcal{M}$ by an FOL fla.

For all $D \subseteq M^n$.

1) D is definable in $\mathcal{M}$ over A

iff D is definable in $\mathcal{M}$ over A by an FOL fla.

2) D is a zeroset in $\mathcal{M}$ over A

iff D is the intersection of countably many sets definable in $\mathcal{M}$ over A by FOL.

Pf of 1). Sps $\mathrm{dist}(x, D)$ is definable in $\mathcal{M}$ over A. Note that $\mathrm{dist}(x, D) = 1 - \chi_D$ where χ_D is the char. funct. of D.

Let φ be $L(A)$-fla s.t. $\forall x \in M^n$.

$$|\mathrm{dist}(x, D) - \varphi^{\mathcal{M}}(x)| \leq \frac{1}{3}.$$

Then $D = \{x \in M^n \mid \varphi^{\mathcal{M}}(x) \leq \frac{1}{2}\}$ and

this's FOL definable in $\mathcal{M}$ over A.

Thus, the definable sets are the same.

Definable functions

A function $f: M^n \to M$ is <u>definable</u> in $\mathcal{M}$ over A if $d(f(x), y)$ on M^{n+1} is a predicate definable in $\mathcal{M}$ over A.

We denote the graph of f by $\mathcal{G}_f \subseteq M^{n+1}$.

<u>Fact</u>: If f is definable in $\mathcal{M}$ over A, then its graph $\mathcal{G}_f$ is definable in $\mathcal{M}$ over A.

pf: $\operatorname{dist}((x,y), \mathcal{G}_f) = \inf_z \max\{d(x,z), d(f(z),y)\}$, where x and z range over M^n, and y ranges over M. $\square$

The converse is not true in general.

Prop D.17 Let $\mathcal{M}$ be a κ-saturated, where κ is uncountable and let $A \subseteq M$ having cardinality $< \kappa$. Let $f: M^n \to M$ be a function. Then, TFAE.

(1) f is definable in $\mathcal{M}$ over A.

(2) $\mathcal{G}_f$ is type-definable in $\mathcal{M}$ over A

(3) $\mathcal{G}_f$ is definable in $\mathcal{M}$ over A

Pf: Prop 9.24. in [BBHU]. $\square$

Prop D.18: Suppose $f: M^n \to M$ is definable in $\mathcal{M}$ over A. Then

(1) If $N \preceq M$, and $A \subseteq N$, then $f(N^n) \subseteq N$

and $f|N^n$ is definable in $\mathcal{N}$ over A.

(2) If $\mathcal{N} \succeq \mathcal{M}$ then there is a function $g: N^n \to N$ such that $g \supseteq f$ and g is definable in $\mathcal{N}$ over A.

Pf: Prop 9.25 in [BBHU]. $\square$

It follows that

1) if given $M^n \xrightarrow[f]{} M \xrightarrow[g]{} M$ and f, g are def, then $g \circ f$ is definable.

2) given $M^n \xrightarrow{f} M \xrightarrow{g} [0,1]$. If f and g are def, then $g \circ f$ is definable.

Omitting types and ω-categoricity

We say that T is <u>$\aleph_0$-categorical</u> if whenever $\mathcal{M}$ and $\mathcal{N}$ are models of T having density charaterer $\aleph_0$, then $\mathcal{M} \cong \mathcal{N}$.

Let p be a type and let $p(\mathcal{M})$ denote the set of all realizations of p in $\mathcal{M}$.

<u>Defn</u>. Let $p \in S_n(T)$. We say that p is <u>principal</u> if for every model $\mathcal{M}$ of T the set $p(\mathcal{M})$ is definable in $\mathcal{M}$ over $\emptyset$.

<u>Lemma D.19</u> Every principal type is realized in every model of T.

pf. 略. $\square$

Prop D.20 Let $p \in S_n(T)$. Then p is principal iff the logic topology and the d-metric topology agree at p.

Pf: Prop 12.4 in [BBHU]. $\square$

Prop D.21 Let $p \in S_n(T)$. Then p is principal iff the ball $\{q \in S_n(T) \mid d(q,p) \leq \varepsilon\}$ has nonempty interior in the logic topology for each $\varepsilon > 0$.

Pf: Prop 12.5 in [BBHU]. $\square$

Thm D.22 (Omitting Types Theorem, local version)
Let T be a complete theory in a countable signature, and let $p \in S_n(T)$. TFAE
(1) p is principal
(2) p is realized in every model of T.

Pf: (1) $\Rightarrow$ (2) Lemma D.19
(2) $\Rightarrow$ (1) consider the contrapositive. Use Prop D.21. $\square$

ω-categorical, a.k.a. separably categorical.

The following theorem is the CFO version of Ryll-Nardzewski theorem.

Thm D.23 Let T be a complete countable theory. TFAE.

(1) T is ω-categorical.

(2) For each n, every type in $S_n(T)$ is principal.

(3) For each n, the metric space $(S_n(T), d)$ is compact.

(4) The logic topology and the d-metric topology coincide.

Pf: Thm 12.10. $\square$

Quantifier elimination

An L-fla $\varphi(x_1, \dots, x_n)$ is approximable in T by quantifier free flas if for every $\varepsilon > 0$

there is a q.f. L-fla $\psi(x_1, \dots, x_n)$ s.t. for all $\mathcal{M} \models T$ and all $a_1, \dots, a_n \in M$, one has

$$|\varphi^{\mathcal{M}}(a_1, \dots, a_n) - \psi^{\mathcal{M}}(a_1, \dots, a_n)| \le \varepsilon.$$

An L-theory T <u>admits quantifier elimination</u> if every L-fla is approximable in T by q.f. flas.

<u>Lemma D.24</u> Suppose that T is an L-theory and that every restricted L-fla of the form $\inf_x \varphi$, with q.f. φ, is approximable in T by q.f. flas. Then T admits quantifier elimination.

Pf: ~~bt~~. $\square$

<u>Prop D.25.</u> Let T be an L-theory. TFAE.

(1) T admits quantifier elimination.

(2) Let $\mathcal{M}, \mathcal{N} \models T$. Then every embedding of a substructure of $\mathcal{M}$ into $\mathcal{N}$ can be extended to an embedding of $\mathcal{M}$ into an elementary extension of $\mathcal{N}$.

pf. [illegible] II

$$\begin{tikzcd}
\mathcal{M} \arrow[r, dashed, "\exists \varphi", "\text{embedding}"'] & \mathcal{N}' \\
A \arrow[u, phantom, "\cup" description] \arrow[r, "\forall f", "\text{embedding}"'] & \mathcal{N} \arrow[u, phantom, "\prec" description]
\end{tikzcd}$$

Stability and independence

3 different approaches to stability in metric structures.

(I) We say T is <u>λ-stable with respect to the discrete metric</u> if for every $M \models T$ and every $A \subseteq M$ of card $\leq \lambda$, the set $S_1(T_A)$ has card $\leq \lambda$.

We say that T is <u>stable</u> w.r.t. the discrete metric if T is λ-stable w.r.t. the discrete metric for some λ.

(II) (Iovino) We say that T is <u>λ-stable</u> if for every $M \models T$ and $A \subseteq M$ of card $\leq \lambda$, there is a subset of $S_1(T_A)$ of card $\leq \lambda$ that is dense in $S_1(T_A)$ with respect to the d-metric.

We say that T is <u>stable</u> if T is λ-stable for some infinite λ.

<u>Theorem D.26</u> A theory T is stable iff T is stable w.r.t. the discrete metric

Pf: Thm 14.6. $\square$

<u>Fact</u>: ω-stable theories have prime models, and ω-stable theories are λ-stable for all infinite λ.

(III). A theory T is stable iff every type over a model M of T is definable over M.