

Day Two

L-conditions $\leftarrow$ new syntax term.

An L-condition E is of the form $\varphi = 0$, where φ is an L-fla. We call E closed if φ is closed, i.e., φ is an L-sentence.

If E is the $L(M)$ -condition

$\varphi(x_1, \dots, x_n) = 0$ and $a_1, \dots, a_n \in M$, we

say E is true of $a_1, \dots, a_n$ in M

and write $M \models E[a_1, \dots, a_n]$ if

$$\varphi^M(a_1, \dots, a_n) = 0.$$

Defn: Let E_i be the L -condition $\varphi_i(x_1, \dots, x_n) = 0$. We say E_1 and E_2 are logically equivalent if for every L -structure M and every $a_1, a_2, \dots, a_n$, we have

$$M \models E_1[a_1, \dots, a_n] \text{ iff } M \models E_2[a_1, \dots, a_n].$$

Some notation:

$\varphi = \psi$ is an abbreviation for the condition

$$|\varphi - \psi| = 0.$$

$(U: [0, 1]^2 \rightarrow [0, 1])$ is a connective
 $(t_1, t_2) \mapsto |t_1 - t_2|$

$$\varphi = r \iff |\varphi - r| = 0.$$

$$\varphi \leq \psi \iff \varphi \dot{-} \psi = 0$$

$$\psi \geq \varphi \iff \varphi - \psi = 0.$$

$$\left(\begin{array}{l} \dot{-} : [0, 1]^2 \rightarrow [0, 1] \text{ is defined by} \\ t_1 \dot{-} t_2 = \begin{cases} t_1 - t_2, & \text{if } t_1 \geq t_2 \\ 0, & \text{else.} \end{cases} \end{array} \right)$$

In $[0, 1]$ -valued logic, $\varphi \leq \psi$ is like

$\varphi \rightarrow \psi$ in FOL. Since from $\varphi \leq r$, we have $\varphi \leq r$ for all $r \in [0, 1]$.

Model theoretic concepts

Fix a signature L for metric structure.

Defn: A theory T is a set of closed L -conditions. We say M is a model

of T and write $M \models T$ if $M \models E$ for every condition E in T .

Let $\text{Mod}_L(T)$ be the collection of all models of T .

The theory of M , denoted by $\text{Th}(M)$, is the set of closed L -conditions that are true in M .

If T is a theory of this form, then T is complete.

We say E is a logical consequence of T and write $T \models E$ if $M \models E$ holds for every model M of T .

Rmk:

(1) models are complete metric spaces

(2) Let M_0 be an L -prestructure s.t.

$\varphi^{M_0} = 0$ for every condition $\varphi = 0$ in T .

Then by Thm A.2, the completion of the canonical quotient of M_0 is a model of T . (M_0 is a premodel.)

Defn:

(1) We say M and N are elementary equivalent, and write $M \equiv N$, if $\sigma^M = \sigma^N$ for all L -sentences σ .

(2) If $M \subseteq N$, we say that M is an elementary substructure of N and write

$M \preceq N$, if whenever $\varphi(x_1, \dots, x_n)$ is an L -f.l.a and $a_1, \dots, a_n \in M$ we have

$$\varphi^M(a_1, \dots, a_n) = \varphi^N(a_1, \dots, a_n).$$

We also say N is an elementary extension of M .

(3) $F: \overset{M}{\bigcup} A \rightarrow N$ is an elementary map if whenever $\varphi(x_1, \dots, x_n)$ is an L -f.l.a and $a_1, \dots, a_n \in \text{dom}(F)$, we have

$$\varphi^M(a_1, \dots, a_n) = \varphi^N(F(a_1), \dots, F(a_n))$$

(4) An elementary embedding of M into N is a function: $M \rightarrow N$ that is an elementary map from M into N .

embedding $\Rightarrow$ injection.

$$\left(d^M(a, b) = d^N(F(a), F(b)) \right)$$

Prop: (1) elementary map is distance preserving, and thus is an embedding.

(2) $M \cong N \Rightarrow M \equiv N$. (HW)

We say S of L -fls is dense w.r.t. logical distance, if for every L -fln $\varphi(x_1, \dots, x_n)$ and every $\varepsilon > 0$, there is $\varphi(x_1, \dots, x_n)$

in S s.t. for every L -structure M and all $a_1, \dots, a_n \in M$, we have

$$|\varphi^M(a_1, \dots, a_n) - \psi^M(a_1, \dots, a_n)| \leq \epsilon.$$

Proposition B.1 (Tarski-Vaught Test for $\preceq$)

Let S be dense w.r.t. logical distance.

Suppose M and N are L -structures with $M \subseteq N$. TFAE.

(1) $M \preceq N$

(2) For every L -fca $\varphi(x_1, \dots, x_n, y)$ in S and all $a \in M^n$,

$$\inf \{ \varphi^N(a, b) \mid b \in N \} = \inf \{ \varphi^N(a, c) \mid c \in M \}$$

(*)

Pf: (1) $\Rightarrow$ (2) $\forall \varphi(x_1, \dots, x_n, y) \in S, \forall a_1, \dots, a_n \in M$

By (1), we have

$$\begin{aligned} \inf \{ \varphi^N(a_1, \dots, a_n, b) \mid b \in N \} &= \left(\inf_y \varphi(a_1, \dots, a_n, y) \right)^N \\ &\stackrel{m \leq N}{=} \left(\inf_y \varphi(a_1, \dots, a_n, y) \right)^m \\ &= \inf \{ \varphi^m(a_1, \dots, a_n, c) \mid c \in M \} \\ &\stackrel{m \leq N}{=} \inf \{ \varphi^N(a_1, \dots, a_n, c) \mid c \in M \} \end{aligned}$$

(2) $\Rightarrow$ (1). First we show (*) holds for all L -fns $\varphi(x_1, \dots, x_n, y)$. $\forall \varepsilon > 0$, take $\varphi(x_1, \dots, x_n, y) \in S$, s.t.

$$\sup_m \sup_{a_1, \dots, a_n \in M} \left| \varphi^m(a_1, \dots, a_n, b) - \varphi^N(a_1, \dots, a_n, b) \right| < \varepsilon$$

Let $a_1, \dots, a_n \in M$. Then we have

$$\begin{aligned}
& \inf \{ \varphi^W(a_1, \dots, a_n, b) \mid b \in M \} \\
& \leq \inf \{ \varphi^W(a_1, \dots, a_n, b) \mid b \in M \} + \varepsilon \\
& = \inf \{ \varphi^W(a_1, \dots, a_n, c) \mid c \in N \} + \varepsilon
\end{aligned}$$

$$\leq \inf \{ \varphi^W(a_1, \dots, a_n, c) \mid c \in N \} + 2\varepsilon$$

$$\begin{aligned}
\text{Let } \varepsilon \rightarrow 0, \quad \inf \{ \varphi^W(a_1, \dots, a_n, b) \mid b \in M \} \\
\leq \inf \{ \varphi^W(a_1, \dots, a_n, c) \mid c \in N \}
\end{aligned}$$

$$\begin{aligned}
\text{Since } M \subseteq N, \quad \inf \{ \varphi^W(a, b) \mid b \in M \} \\
\geq \inf \{ \varphi^W(a, c) \mid c \in N \}.
\end{aligned}$$

Thus, (*) holds for all L -fls φ .

Then by induction on the complexity of φ and b) , we have $\varphi^M(a_1, \dots, a_n) = \varphi^W(a_1, \dots, a_n)$

for all $a_1, \dots, a_n \in M$.

□

Ultraproducts and compactness

(We never mention completeness theorem " $\vdash$ " Check Ben Yaacov's paper)

Quick review of ultrafilters

Let I be a nonempty set. A filter on I

is a collection F of subsets of I satisfying

(i) $\emptyset \notin F$ and $I \in F$

(ii) for all $A, B \in F$, $A \cap B \in F$.

(iii) for all $A \in F$ and $B \subseteq I$, if $A \subseteq B$ then $B \in F$.

Elements in F are big elements.
 $\uparrow$
filter.

Example: $I = \mathbb{N}$, $F = \{A \mid |\mathbb{N} \setminus A| < \infty\}$
 F is a Frechét filter. $\uparrow$ cofinite

A filter F is an ultrafilter if it is maximal under $\subseteq$ among filters on I .

F is principal if there is a subset $A \subseteq I$ s.t. F is exactly the collection of all sets B that satisfy $A \subseteq B \subseteq I$.

non-principal is also called as free

Defn: S is a collection of I . We say that S has finite intersection property (FIP), if $\forall n \in \mathbb{N} \forall$ finite subcollections

$\{A_1, \dots, A_n\}$ of S , $A_1 \cap \dots \cap A_n \neq \emptyset$.

Lemma B.2 Let I be a nonempty set and let S be a collection of subsets of I . There exists a filter F on I which contains S iff S has the FIP.

pf: HW. □

Def: The smallest filter on $I \supseteq S$ is called the filter generated by S .

Lemma B.3 Let F be a filter on a nonempty set I . Then F is an ultrafilter iff $\forall A \subseteq I$, either $A \in F$ or $A^c \in F$.

Proof: HW. $\square$

Rmk: principal ultrafilters are trivial.

Thm B.4: Let I be a nonempty set.

Then every filter on I is contained in an ultrafilter on I .

proof: using Zorn's lemma. $\square$

Corollary B.5: Let I be a nonempty set and let S be a collection of subsets of I .

If S has the FIP, then there is an ultrafilter on I that contains S . $\square$

Ultraproducts of first order structures

Fix a first order signature L .

Let I be a nonempty set and let $\mathcal{U}$ be a fixed ultrafilter on I . Consider an I -indexed family of L -structures A_i .

Let $A := \prod_{i \in I} A_i$ be the cartesian product of the sets A_i . Let $f, g \in A$.

We define a relation on A

$$f \sim g \text{ iff } \{i \mid f(i) = g(i)\} \in \mathcal{U}.$$

Lemma B.6: The relation $\sim$ is an equivalence relation on $A = \prod_{i \in I} A_i$.

Proof: HW.

□

Then $A/\sim$ is the ultraproducts of the set A_i with respect to the ultrafilter u on I .

We let $\prod_u A_i$ denote $A/\sim$, the

collection of all equivalence classes

$$\{ f/\sim \mid f \in \prod_{i \in I} A_i \}$$

Defn of the ultraproduct:

The ultraproduct $\prod_u A_i$ is defined to be the L -structure.

(i) the universe of $\prod_u A_i$ is $\prod_u A_i$

(i') for each constant c in L .

Define $f \in A$ by $f(i) = c^{A_i} \forall i \in I$.

$$c^{\prod A_i} = f/\sim$$

(ii') for each predicate P in L .

$$P^{\prod A_i}(f_1/\sim, \dots, f_n/\sim)$$

$$\text{iff } \{i \in I \mid P^{A_i}(f_1(i), \dots, f_n(i))\} \in U$$

(i') for each function F in L .

$$F^{\prod A_i}(f_1/\sim, \dots, f_n/\sim) = f/\sim, \text{ where}$$

$$f \in A \text{ is defined by } f(i) = F^{A_i}(f_1(i), \dots, f_n(i)).$$

Need to check it's well-defined.

An ultrapower of A is an ultraproduct $\prod_u A_i$ with $A_i = A$ for all $i \in I$.

Los Theorem B.7 (Fundamental Thm of Ultraproducts)

For every L -fba $\varphi(x_1, \dots, x_n)$ and every $f/\sim = (f_1/\sim, \dots, f_n/\sim)$, we have

$$\prod_u A_i \models \varphi[f/\sim] \quad \text{iff}$$

$$\{i \in I \mid A_i \models \varphi[f_1(i), \dots, f_n(i)]\} \in \mathcal{U}.$$

Proof: HW.

□

Corollary B.8: If σ is an L -sentence, then $\prod_{i \in I} A_i \models \sigma$ iff $\{i \in I \mid A_i \models \sigma\} \in \mathcal{U}$. $\square$

Back to CPO

Let X be a topological space and let $(x_i)_{i \in I}$ be a family of elements of X . If $\mathcal{D}$ is an ultrafilter on I and $x \in X$, we write

$$\lim_{i \in \mathcal{D}} x_i = x \quad \Leftarrow \text{ultra limit}$$

and say x is the $\mathcal{D}$ -limit of $(x_i)_{i \in I}$ if $\forall \text{ open } U \ni x, \{i \in I \mid x_i \in U\} \in \mathcal{D}$.

✓ e.g. $X = [0, 1]$.

Fact: X is a compact Hausdorff space
iff for every family $(x_i)_{i \in I}$ in X and every
ultrafilter D on I the D -limit of $(x_i)_{i \in I}$
exists and is unique.

ultra limits $\longleftrightarrow$ Semantics of CPO.

Lemma B-9: Suppose X, X' are topological
spaces and $F: X \rightarrow X'$ is continuous. For
every family $(x_i)_{i \in I}$ from X and every
ultrafilter D on I , we have

$$\lim_{i, D} x_i = x \Rightarrow \lim_{i, D} F(x_i) = F(x).$$

where the ultralimits are taken in X and X' respectively.

Pf: Take open $U \ni F(x)$ in X' .

Since F is cont., $F^{-1}(U)$ is open in X , and $F^{-1}(U) \ni x$. If x is the D -limit of $(x_i)_{i \in I}$, there is $A \in D$ s.t. $x_i \in F^{-1}(U)$ for all $i \in A$ and thus $F(x_i) \in U$. $\square$

Ultraproducts of metric spaces

Let $(M_i, d_i) \mid i \in I$ be a family of bddd metric spaces with diameter $\leq K$.

Let D be an ultrafilter on I .

Define d on $\prod_{i \in I} M_i$ by

$$d(x, y) = \lim_{\delta, D} d_i(x_i, y_i), \in [0, K]$$

where $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$.

Check: d is a pseudometric on $\prod_{i \in I} M_i$.

For $x, y \in \prod_{i \in I} M_i$, define $x \sim_D y$ iff $d(x, y) = 0$.

Then $\sim_D$ is an equivalence relation, so we may define.

$$\left(\prod_{i \in I} M_i \right)_D = \left(\prod_{i \in I} M_i \right) / \sim_D.$$

Later, we will see it's complete.

The pseudometric d on $\prod_{i \in I} M_i$ induces

a metric d on $(\prod_{i \in I} M_i)_D$.

The space $((\prod_{i \in I} M_i)_D, d)$ is the D -ultraproduct of $((M_i, d_i) / i \in I)$.

We denote $(x_i)_{i \in I} / \sim_D$ by $(x_i)_{i \in I}_D$.

If $(M_i, d_i) = (M, d) \forall i \in I$, the space $(\prod_{i \in I} M_i)_D$ is called the ultrapower of M and denoted by $(M)_D$.

$\mathcal{T}: M \rightarrow (M)_D$, where $x_i = x \forall i \in I$
 $x \mapsto (x_i)_{i \in I}_D$ is a diagonal embedding
is an isometric embedding

If (M, d) is compact, then it is easy to show $((x_i)_{i \in I})_D = T(x)$, i.e., the diagonal embedding is surjective.

Fact: Every ultrapower of a closed bdd interval may be canonically identified with the interval itself, e.g., $([0, 1])_D = [0, 1]$.

Prop B.10: Let $((M_i, d_i) \mid i \in I)$ be a family of complete, uniformly bdd metric space. Let D be an ultrafilter on I and let (M, d) be the D -ultraproduct

of $(M_i, d_i) (i \in I)$, The metric space (M, d) is complete.

Pf: Let $(x^k)_{k \geq 1}$ be a Cauchy seq. in (M, d) . WLOG, WMA $d(x^k, x^{k+1}) < \frac{1}{2^k}$ holds for all $k \geq 1$. We want to show it has a limit.

For each $k \geq 1$, let x^k be represented by the family $(x_i^k)_{i \in I}$.

For each $m \geq 1$, let $A_m = \{i \in I \mid d_i(x_i^k, x_i^{k+1}) < \frac{1}{2^k} \forall k \leq m\}$

Note: $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq \dots$

We define a family $(y_i)_{i \in I}$ as follows.

If $i \in A_1$, then we take y_i arbitrarily.

If $i \in A_m \setminus A_{m+1}$ for some $m \geq 1$, then

we set $y_i = x_i^{m+1}$.

If $i \in A_m \forall m \geq 1$, then $(x_i^m)_{m \geq 1}$ is a

Cauchy seq, we take y_i to be its limit.

Then for each $m \geq 1$, each $i \in A_m$, we have

$$d_i(x_i^m, y_i) \leq 2^{-m+1}.$$

It follows that $((y_i)_{i \in I})_D$ is the limit
of $(x^k)_{k \geq 1}$ in (M, d) . □

Ultraproducts of functions

Suppose $(M_i, d_i) (i \in I)$ and $(M'_i, d'_i) (i \in I)$ are families of metric spaces, with diameter $\leq K$.

Fix $n \geq 1$ and sps $f_i: M_i^n \rightarrow M'_i$ is uniformly continuous function for each $i \in I$.

Moreover, there is a function $\delta: (0, 1] \rightarrow (0, 1]$ is a modulus of uniform continuity for all f_i .

Given an ultrafilter D on I . We define

$$\left(\prod_{i \in I} f_i \right)_D : \left(\prod_{i \in I} M_i \right)_D^n \rightarrow \left(\prod_{i \in I} M'_i \right)_D$$

as follows:

$$\left(\prod_{i \in I} f_i \right)_D \left((x_i^1)_{i \in I} \right)_D, \dots, (x_i^n)_{i \in I} \right)_D \\ = \left((f_i(x_i^1, \dots, x_i^n))_{i \in I} \right)_D,$$

where $(x_i^k)_{i \in I} \in \prod_{i \in I} M_i$.

Claim: $\left(\prod_{i \in I} f_i \right)_D$ is a uniformly cont.
function that also has Δ as its
modulus of uniform continuity.

Pf: Sps $n=1$, $\forall \varepsilon > 0$, Sps $d\left(\left((x_i)_{i \in I}\right)_D, \left((y_i)_{i \in I}\right)_D\right) < \Delta(\varepsilon)$

Then $\exists A \in D$ s.t. $\forall i \in A$, $d_i(x_i, y_i) < \Delta(\varepsilon)$.

Since Δ is a modulus of u.c. for all f_i .
Then $d_i(f_i(x_i), f_i(y_i)) \leq \varepsilon$ for all $i \in A$.

Hence, $d'((f(x_i))_{i \in I})_D, ((f(y_i))_{i \in I})_D \leq \varepsilon$.

This shows that $(\prod_{i \in I} f_i)_D$ is well-defined and has Δ as a modulus of u.c. $\square$

Ultraproducts of L-structures

Let $(M_i | i \in I)$ be a family of L-structures and let D be an ultrafilter on I .

We define the D -ultraproduct of the family $(M_i | i \in I)$ of L-structures to be the L-structure M that is specified as follows.

(i) the universe of M is $M = (\prod_{i \in I} M_i)_D$

(i) for each predicate p of L .

$$p^M = \left(\prod_{i \in I} p^{\mathcal{M}_i} \right)_D : M^n \rightarrow \{0, 1\}$$

(ii) for each function f of L . $\left(\left(\{0, 1\} \right)_D \right)_D = \{0, 1\}$

$$f^M = \left(\prod_{i \in I} f^{\mathcal{M}_i} \right)_D : M^n \rightarrow M.$$

(iv) for each constant c of L .

$$c^M = (c^{\mathcal{M}_i})_{i \in I}_D.$$

The definition is well-defined (check!).

We call M be D -ultraproduct of the family $(\mathcal{M}_i | i \in I)$ and denoted by

$$M = \left(\prod_{i \in I} \mathcal{M}_i \right)_D.$$

D-ultrapower. $(\mathcal{M})_D$.

Los Theorem B.11 Let $\varphi(x_1, \dots, x_n)$ be an L -f.l.a. If $a_k = ((a_i^k)_{i \in I})_D$ are elements of $\mathcal{M}$ for $k=1, \dots, n$, then

$$\varphi^{\mathcal{M}}(a_1, \dots, a_n) = \lim_{i \in D} \varphi^{\mathcal{M}_i}(a_i^1, \dots, a_i^n).$$

Pf: Induction on the complexity of φ and ultrafilters. $\square$

Corollary B.12: If $\mathcal{M}$ is an L -structure and $T: \mathcal{M} \rightarrow (\mathcal{M})_D$ is the diagonal embedding, then T is an elementary embedding of $\mathcal{M}$ into $(\mathcal{M})_D$. $\square$

Corollary B.13: If M and N are L -structs and they have isomorphic ultrapowers, then $M \equiv N$.

Pf: Note $M \equiv (M)_D$. $\square$

Keisler-Shelah Thm B.14 If M and N are L -structures and $M \equiv N$, then there is an ultrafilter D such that $(M)_D \cong (N)_D$.

Compactness Theorem

Compactness Thm B.15 Let T be an L -theory and $\mathcal{I}$ a class of L -structures. Assume that T is finitely satisfiable in $\mathcal{I}$. Then there is an ultraproduct of structure from $\mathcal{I}$ that is a model of T .

Pf: Let Λ be the set of finite subsets of T . Let $\lambda \in \Lambda$, and write $\lambda = \{E_1, \dots, E_n\}$. By assumption, $\exists M_\lambda \in \mathcal{L}$ s.t. $M_\lambda \models \lambda$.

For each $E \in T$, let $S(E)$ be the set of all $\lambda \in \Lambda$ s.t. $E \in \lambda$. Note that the collection of sets $\{S(E) \mid E \in T\}$ has the FIP. (Check!)

Hence, there is a ultrafilter $\mathcal{D}$ on Λ that contains $S(E)$.

$$\text{Let } \mathcal{M} = \left(\prod_{\lambda \in \Lambda} M_\lambda \right)_{\mathcal{D}}.$$

Note that if $\lambda \in S(E)$, then $M_\lambda \models E$.
By Los Thm, $\mathcal{M} \models E$ for every $E \in T$.

ie., $\mathcal{M} \models T$.

□