

Day One

Continuous First Order Logic

a.k.a. continuous logic, continuous model theory, model theory for metric structures.

References:

[BBHU] Ben Yaacov, Berenstein, Henson, Usvyatsov, Model theory for metric structures, in: Model Theory with Applications to Algebra and Analysis,

LMS Lecture Note Series, 350, 2008
315 - 427.

[CBU] Ben Yaacov and Usvyatsov, Continuous first order logic and local stability, Trans. AMS, 362 (2010), 5213 - 5259.

Some History:

Logic

1930s

Compactness Thm
Ultraproducts, saturation

Mathematics

1960s

- ① Ax-Kochen, Ershov
Diophantine problems
over local fields
- ② Abraham Robinson
Nonstandard analysis

1970s

Shelah - Classification Theory
Stability theory

1980s

O-minimal theory

2000s

Continuous Logic

← 7 ...

1996

Hrushovski
Mordell-Lang
conjecture

2011

Pila
André-Oort Conj.

2020s

major breakthrough

Background

1960s Applications of Ultraproducts to
Banach spaces

Krivine. "stable"-like results.

1976, Henson, Nonstandard hulls of Banach spaces, Israel J. Math, 1976.

"Henson's logic" positive bdded f-las with an approximate semantics.

Later, 2002 with Iovino. Ultraproducts in analysis, LMS LNS, 262, 2002.

2003, Ben Yaacov, Compact abstract theories (cat)

Positive model theory and compact abstract theories, JML, 2003.

Ben Yaacov, Berenstein, Henson, Usvyatsov,
≈ 2005 metric structures.

many valued logic : Łukasiewicz
Chang-Keisler
张晨钟

{ truth values : $[0, 1]$.
quantifiers : $\inf, \sup$
equality = : metric $d(\cdot, \cdot)$

Analogy between CFO and FOL.

Compactness Thm $\rightsquigarrow$ saturation

Löwenheim-Skolem Thms

Omitting Types Thm.

Quantifier Elimination

Stability theory. etc

Metric structures and signatures

Let (M, d) be a complete, bdded metric space.

d is a metric on M .

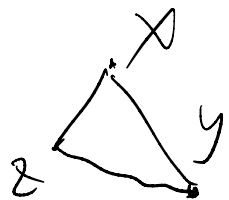
$$d: M \times M \rightarrow \mathbb{R}^{\geq 0}$$

$$(x, y) \mapsto d(x, y)$$

$$(1) \quad d(x, y) = d(y, x)$$

$$(2) \quad d(x, y) \geq 0 \text{ and } d(x, y) = 0 \text{ iff } x = y$$

$$(3) \quad d(x, y) \leq d(x, z) + d(z, y)$$



(M, d) is bdded if $\exists K > 0, \forall x \neq y \in M,$
 $d(x, y) < K$.

or. $\text{diam}(M, d) := \sup_{x, y \in M} d(x, y) < K$.

(M, d) is complete if every Cauchy

Sequence has a limit in M .

$\{a_n\}_{n \in \mathbb{N}} \subseteq M$, is a Cauchy sequence if $\forall \varepsilon > 0 \exists N \forall m, n > N d(a_m, a_n) < \varepsilon$.

Ex: $([0, 1], d)$, where $d(x, y) = |x - y|$,
is a complete mdded metric space.

A predicate on M is a uniformly continuous
function: $M^n \rightarrow [a, b] \subseteq \mathbb{R}$ for some $n \geq 1$.

$p: M^n \rightarrow [0, 1]$ is uniformly continuous

if $\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in M^n (d(x, y) < \delta$
 $\rightarrow |p(x) - p(y)| < \varepsilon)$

A function on M is a uniformly continuous function $: M^n \rightarrow M$ for some $n \geq 1$.
 $\uparrow$
 arity.

For simplicity, (M, d) is added by 1 ~~T~~
 predicates have values on $\{0, 1\}$. F
 $\uparrow$
 truth values

Rmk: FOL predicates: $M^n \rightarrow \{0, 1\}$
 CFO predicates: $M^n \rightarrow [0, 1]$

A metric structure M based on (M, d) consists of a family $(R_i | i \in I)$ of predicates on M , a family $(F_j | j \in J)$ of functions on M , and a family $(a_k | k \in K)$ of distinguished

elements of M .

We denote a metric structure as

$$\mathcal{M} = (\mathcal{M}, R_i, F_j, a_k \mid i \in I, j \in J, k \in K)$$

Key restrictions :

- 1) complete bdd.
- 2) bdd ed intervals $\leftarrow R_i$
- 3) unif. cont. $\leftarrow R_i$
 $\nwarrow F_j$

Examples:

- (1) A complete bdded metric space (M, d) with no additional structures.
- (2) A first-order structure M .
- Define a discrete metric on M by
- $$d(a, b) = \begin{cases} 1, & \text{if } a \neq b \in M \\ 0, & \text{if } a = b \in M. \end{cases}$$

predicates taking values in $\{0, 1\}$

Then (M, d) is a metric structure.

Thus, CFO is a generalization of FOL.

(3) Probability algebras are boolean algebras of events in probability space. We will discuss it further later.

Syntax

Signatures

To each metric structure

$$M = (M, R_i, F_j, a_k \mid i \in I, j \in J, k \in K)$$

we associate a Signature

$$L = \{P_i, f_j, c_k \mid i \in I, j \in J, k \in K\}$$

where each p_i is a predicate symbol corresponding to predicate P_i on M ,
each f_j is a function symbol corresponding to function F_j on M ,
and each c_k is a constant symbol corresponding to constant a_k of M .

We also associate to each symbol its arity like in FOL.

Moreover, in CFO, we associate to each predicate symbol P a closed bdded interval I_P (usually, $I_P = [0, 1]$) and a modulus of uniform continuity Δ_P .

i.e., a function $\Delta_p : (0, 1] \rightarrow (0, 1]$ satisfying
 $\varepsilon \mapsto \Delta_p(\varepsilon) = \delta$
 $\forall \varepsilon > 0 \forall x, y \in M^n$, if $d(x, y) < \Delta_p(\varepsilon)$ then
 $|p^M(x) - p^M(y)| < \varepsilon$.

This Δ_p does NOT depend on M .

We associate to each function a
modulus of uniform continuity Δ_f ,
i.e., a function $\Delta_f : (0, 1] \rightarrow (0, 1]$ satisfying
 $\forall \varepsilon > 0 \forall x, y \in M^n$, if $d(x, y) < \Delta_f(\varepsilon)$ then
 $d(f^M(x), f^M(y)) < \varepsilon$.

Finally, L provides $D_L > 0$ which is a
bound on the diameter of (M, d)

For simplicity, $D_L = 1$ and $I_p = [0, 1]$.

We say M an L -structure.

Suppose M and N are L -structures.

An embedding from M to N is a metric space isometry $T: (M, d^M) \rightarrow (N, d^N)$ such that

- 1) isometry: $\forall x, y \in M, d^N(T(x), T(y)) = d^M(x, y)$.
- 2) for each n -ary predicate symbol P of L
 $\forall a_1, \dots, a_n \in M, P^N(T(a_1), \dots, T(a_n)) = P^M(a_1, \dots, a_n)$.
- 3) for each n -ary function symbol f of L
 $\forall a_1, \dots, a_n \in M, f^N(T(a_1), \dots, T(a_n)) = T(f^M(a_1, \dots, a_n))$.
- 3) for each constant symbol c of L .
 $c^N = T(c^M)$.

An isomorphism is a surjective embedding.

We say that M and N are isomorphic, and write $M \cong N$ if there exists an isomorphism between M and N .

An automorphism of M is an isomorphism between M and itself.

If $M \subseteq N$ and the inclusion map $i: M \hookrightarrow N$ is an embedding of M into N , we say that M is a substructure of N , and write $M \subseteq N$.

More on modulus of uniform continuity

See [BBHU] appendix to Section 2 on pages 322-327.
There are some technical lemmas.

$$d \text{ on } M^n$$

$$d((x_1, \dots, x_n), (y_1, \dots, y_n))$$

$$= \max \{d(x_1, y_1), \dots, d(x_n, y_n)\}$$

Formulas

Fix a signature L for metric structures

Symbols of L

nonlogical symbols { predicates
functions
constants

Logical symbols { $d \leftarrow$ binary predicate
 $V_L \leftarrow$ an ∞ set of variables
 $u: [0,1]^n \rightarrow [0,1]$ continuous
(connectives)
 $\sup, \inf$ (quantifiers)

FOL $\vee, \wedge, \neg$

Connectives: $\{0, 1\}^n \rightarrow \{0, 1\}$

CFO $\neg, \frac{1}{2}$

connectives: $[0, 1]^n \rightarrow [0, 1]$

continuous $\Rightarrow$ unif. cont.

The cardinality of L , denoted by $\text{Card}(L)$, is the smallest infinite cardinal $\geq \#\{\text{nonlogical symbols}\}$.

Terms of L $\leftarrow$ using functions & constants

Inductively, exactly as in FOL

① variable and constant symbols

② $f(t_1, \dots, t_n)$ is an L -term, where f is n -ary function, $t_1, \dots, t_n$ are L -terms.

Atomic formulas of L

- ① $P(t_1, \dots, t_n)$, where P is an n -ary predicate symbol and $t_1, \dots, t_n$ are L -terms
- ② $d(t_1, t_2)$, where t_1 and t_2 are L -terms
↑ like " $=$ " in FOL

Formulas of L

The class of L -flas is the smallest class of expressions satisfying the following requirements

- ① atomic flas are L -flas
- ② If $u: [0, 1]^n \rightarrow [0, 1]$ is continuous, and $\varphi_1, \dots, \varphi_n$ are L -flas, then $u(\varphi_1, \dots, \varphi_n)$ is an L -fla.

③ If φ is an L-fla and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are L-flas.

Rmk: The defn is not a good one.

- (1) too general, unctbly many continuous functions.
a dense subset will be fine.
- (2) too restrict, in order to develop a good notion of "definability".
need flas closed under certain kinds of limits.

An L-fla is quantifier free if neither $\sup_x$ nor $\inf_x$ are used.

Similarly, in FOL, we have notions of subfla, free and bound occurrences of variables.

An L-sentence is an L-fcn that has no free variables.

We write $t(x_1, \dots, x_n)$ or $\varphi(x_1, \dots, x_n)$ to indicate that its free variables are among $x_1, \dots, x_n$.

Semantics prestructure:

Example: Let D_0 denote the set of repeating decimals. Then (D_0, d_0) is a pseudometric space. Because, $d_0(0.\dot{9}, 1) = 0$ but " $0.\dot{9} \neq 1$ ". Consider its quotient $(D, d) = (D_0, d_0) / \sim$, where $x \sim y$ if $d_0(x, y) = 0$.

$$1/\sqrt{2} = 1.199\dots$$

Then (D, d) is a metric space, but it is not complete. Actually, $(D, d) = (\mathbb{Q}, d)$.

Take its completion, we get $(\bar{D}, \bar{d}) = (\mathbb{R}, d)$.

$\text{pseudometric space} \rightarrow \text{metric space} \rightarrow \text{complete metric space}$

Fix a signature L . Let (M_0, d_0) be a pseudometric space, satisfying $\text{diam}(M_0, d_0) \leq 1$.

An L -prestructure M_0 based on (M_0, d_0) is a structure satisfying

(1) for each predicate symbol p of L .

$\mu^{M_0}_p : M_0^n \rightarrow I_p$ has Δ_p as a modulus of unif. cont.

(2) for each function symbol f of L
 $f^{M_0} : M_0^n \rightarrow M_0$ has Δ_f as a modulus
of unif. cont.

(3) for each constant symbol c of L
 $c^{M_0} \in M_0$

Given an L -prestructure M_0 , we define
its quotient prestructure as follows

Let $(M, d) = (M_0, d_0) / \sim$, where
 $x \sim y$ iff $d_0(x, y) = 0$.

Let $\pi : M_0 \rightarrow M$ be the quotient map.

Then
(4) for each predicate symbol P of L .
Define $P^M : M^n \rightarrow I_P$ by

$$p^M(\pi(x_1), \dots, \pi(x_n)) = p^{M_0}(x_1, \dots, x_n) \quad \forall x_1, \dots, x_n \in M_0.$$

(2) for each function symbol f of L .

Define $f^M: M^n \rightarrow M$ by

$$f^M(\pi(x_1), \dots, \pi(x_n)) = \pi(f^{M_0}(x_1, \dots, x_n))$$

$$\forall x_1, \dots, x_n \in M_0$$

(3) for each constant symbol c of L .

$$\text{Define } c^M = \pi(c^{M_0}).$$

Clearly, (1) $\text{diam}(M, d) = \text{diam}(M_0, d_0)$

(2) p^M is well-defined and has

Δ_p as its modulus of u.c.

(3) f^M is well-defined and has

Δ_f as its modulus of u.c.

Thus, (M, d) is an L -pre structure based on a possibly incomplete metric space.

Finally, we take a completion of M , denoted by L -structure N .

(1) for each p .

Define $p^N: N^n \rightarrow I_p$ as the unique extension of $p^M: M^n \rightarrow I_p$ with the same Δ_p (check!)

(2) for each f .

$f^N: N^n \rightarrow N$ is the unique extension of $f^M: M^n \rightarrow M$ with the same Δ_f .

(3) for each constant c , $c^N = c^M$.

Then N is an L -structure.

↓
complete bdd metric space

Semantics

Let $\mathcal{M}$ be an L -prestructure, and let $A \subseteq M$. We extend L to a signature $L(A)$ by adding a new constant symbol $c(a)$ to L for each $a \in A$.

$$(c(a))^{\mathcal{M}} = a \text{ for each } a \in A.$$

We call $c(a)$ the name of a in $L(A)$.

Consider $L(\mathcal{M})$ -terms $t(x_1, \dots, x_n)$.

Define exactly as in FOL.

$$t^{\mathcal{M}} : M^n \rightarrow M$$

↑ interpretation of t in $\mathcal{M}$.

Key defn of semantics in CFO

- (1) $(d(t_1, t_2))^M = d^M(t_1^M, t_2^M)$ for all t_1, t_2
- (2) $(p(t_1, \dots, t_n))^M = p^M(t_1^M, \dots, t_n^M)$ for all
n-ary predicate symbol p and all $t_1, \dots, t_n$.
- (3) for all $L(M)$ -sentences $\sigma_1, \dots, \sigma_n$ and
all continuous functions $U: [0, 1]^n \rightarrow [0, 1]$
 $(U(\sigma_1, \dots, \sigma_n))^M = U(\sigma_1^M, \dots, \sigma_n^M).$
- (4) for all $L(M)$ -fals $\varphi(x)$
 $(\sup_x \varphi(x))^M = \sup_{a \in M} (\varphi(a))^M \in [0, 1]$
 $(\inf_x \varphi(x))^M = \inf_{a \in M} (\varphi(a))^M \in [0, 1]$

Defn: Given $L(M)$ -f/a $\varphi(x_1, \dots, x_n)$, we let φ^M denote the function $M^n \rightarrow [0, 1]$ defined by

$$\varphi^M(a_1, \dots, a_n) = (\varphi(a_1, \dots, a_n))^M.$$

Fact: φ^M is a uniformly continuous function.

Thm A.1: Let $t(x_1, \dots, x_n)$ be an L -term and $\varphi(x_1, \dots, x_n)$ an L -f/a. Then there exists functions Δ_t and $\Delta_\varphi : (0, 1) \rightarrow (0, 1)$ such that for every L -prestructure M , Δ_t is a modulus of uniform continuity for the function $t^M : M^n \rightarrow M$ and Δ_φ is a modulus of uniform continuity for the predicate $\varphi^M : M^n \rightarrow [0, 1]$.

Pf: Induction on terms and f/as.

Technical lemmas on u.c.

□

Thm A.2

pseudometric space $\rightsquigarrow$ quotient $\rightarrow$ completion
 $(M_0, d_0) \xrightarrow{\pi} (M, d) \rightarrow (N, d)$

Let $t(x_1, \dots, x_n)$ be an L -term and $\varphi(x_1, \dots, x_n)$ be an L -f.a. Then

- (1) $t^M(\pi(a_1), \dots, \pi(a_n)) = t^{M_0}(a_1, \dots, a_n) \forall a_1, \dots, a_n \in M_0$
- (2) $t^N(b_1, \dots, b_n) = t^M(b_1, \dots, b_n) \forall b_1, \dots, b_n \in M$
- (3) $\varphi^M(\pi(a_1), \dots, \pi(a_n)) = \varphi^{M_0}(a_1, \dots, a_n) \forall a_1, \dots, a_n \in M_0$
- (4) $\varphi^N(b_1, \dots, b_n) = \varphi^M(b_1, \dots, b_n) \forall b_1, \dots, b_n \in M$.

$\square$ pf: Induction on terms and then on f.a.s.

In (3), key step is that π is surjective.

In (4), key step is that φ^N is continuous and M is dense in N . $\square$

Logical equivalence

Two L -fms $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are logically equivalent if

$$\varphi^M(a_1, \dots, a_n) = \psi^M(a_1, \dots, a_n)$$

for every L -structure M and every $a_1, \dots, a_n \in M$.

The logical distance d_L between φ and ψ is

$$d_L(\varphi, \psi) = \sup_M \sup_{a_1, \dots, a_n \in M} |\varphi^M(a_1, \dots, a_n) - \psi^M(a_1, \dots, a_n)|$$

Prop : (1) This defines a pseudometric

(2) $d_L(\varphi, \psi) = 0$ iff $\varphi \sim_L \psi$.

Size of the space of L-flas

The space of L-flas is too big.

density character : smallest dense

subset w.r.t. logical distance

By Weierstrass Thm, there is a countable set of functions $[0,1]^2 \rightarrow [0,1]$, that is dense in the set of all cont. functs. w.r.t. sup-distance. We may use this countable set of functions $[0,1]^2 \rightarrow [0,1]$ to build connectives. Then

- (1) the total number of constructed flas is $\leq \text{Card}(\mathbb{Q})$
- (2) every L-fla can be approximated arbitrarily closely in logical distance by a formula constructed using restricted connectives.