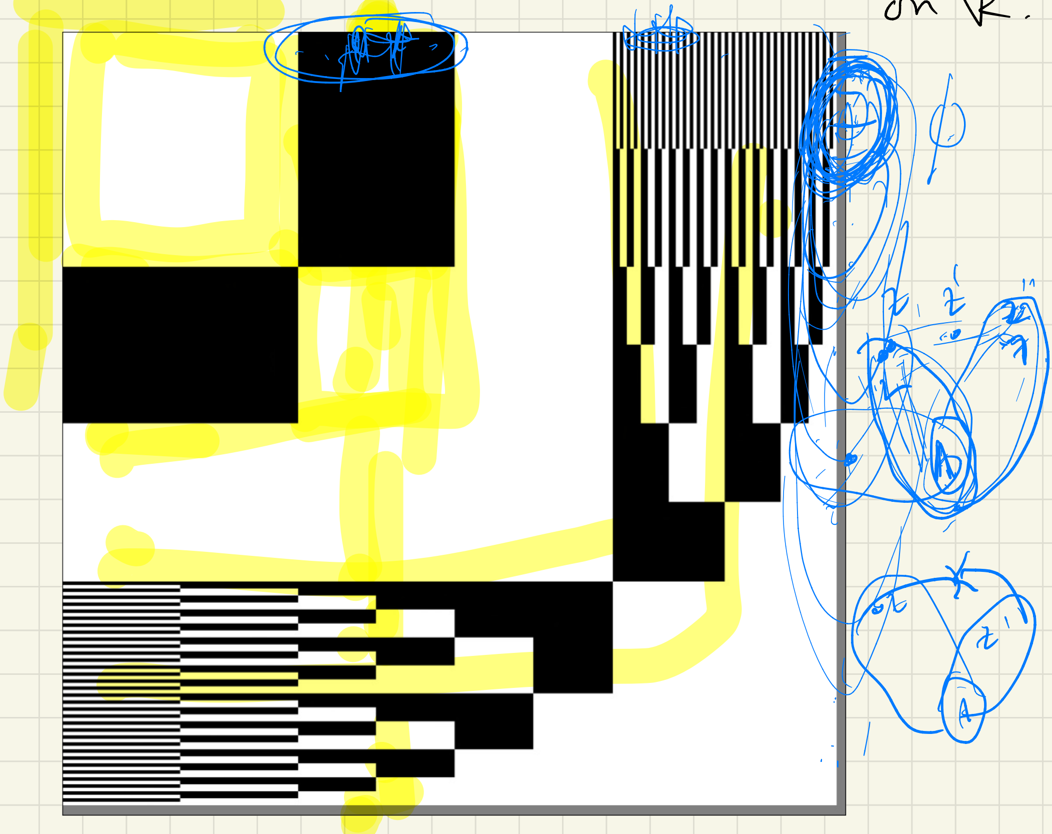
# PROBABILISTIC CONSTRUCTIONS IN MODEL THEORY

## DAY 4

Borel graph producing inv. measure on $\mathbb{R}$.



<u>Recall</u>: $M$ a Fraïssé limit in a finite relational language:

$M$ admits an invariant measure (has exchangeable construction).

$\iff$ $M$ has strong amalgamation.

# A turn towards pure logic

$\mathcal{L}_{\omega_1,\omega}$: logic like first-order but we allow countable infinite $\bigwedge$s and $\bigvee$s.

<u>Notation</u>: Given $\mathcal{L}_{\omega_1,\omega}$-formula $\varphi(\bar{x})$ in $\mathcal{L}$, and $\bar{a} \in \mathbb{N}$, $|\bar{a}| = |\bar{x}|$, say <u>extent</u> of $\varphi(\bar{a})$ is: $[\![ \varphi(\bar{a}) ]\!] := \{ \mathcal{M} \in Str_{\mathcal{L}} : \mathcal{M} \models \varphi(\bar{a}) \}$

Extent of any sentence is a Borel subset of $Str_{\mathcal{L}}$.

$\mu$ invariant:

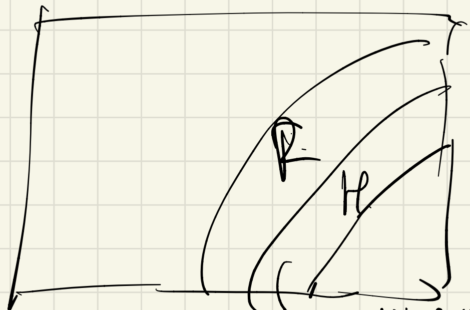
$\mu([\![ \varphi(\bar{a}) ]\!]) = \mu([\![ \varphi(\bar{b}) ]\!])$


$\mathcal{L}$: countable language.

## Scott sentence:

For any countable structure $\mathcal{M}$, there is an $L_{\omega_1\omega}$-sentence $\varphi_{\mathcal{M}}$ such that: for any countable infinite $\mathcal{N}$, $\mathcal{N} \models \varphi_{\mathcal{M}}$ iff $\mathcal{N} \cong \mathcal{M}$

$\varphi_{\mathcal{M}}$ is called a Scott sentence for $\mathcal{M}$.




So asking for an invariant measure concentrated on $[\![\varphi]\!]$ for some arbitrary $L_{\omega_1\omega}$-sentence $\varphi$ is a natural generalisation of asking for an exchangeable construction of a particular structure.

This more general question also has an answer:

$\boxed{\text{Theorem}}$: (Ackerman–Freer–P. 2017):

$\mathcal{L}$ a countable language, $T$ a countable $\mathcal{L}_{\omega_1,\omega}(\mathcal{L})$-theory.

TFAE:

1. There is an invariant measure on $\mathrm{Str}_{\mathcal{L}}$ concentrated on $[\![ \bigwedge_{\varphi \in T} \varphi ]\!]$.

2. There is a countable fragment $F$ of $\mathcal{L}_{\omega_1,\omega}(\mathcal{L})$ and a complete $F$-theory $\Sigma$ such that $T \subseteq \Sigma$ and $\Sigma$ has syntactic trivial definable closure.

$T$ is an $F$-theory if $T$ has a model + for any $\phi \in F$, $\phi$ or $\neg\phi \in T$ ded

$F$ fragment: contains all atomics + closed under subformulas, finite Boolean combinations, quantification, substitution of free variables.

$\Sigma$ has <u>syntactic trivial definable closure</u>

if there is **no** formula $\varphi(\bar{x}, y)$ in $F$, with say $|\bar{x}| = n$, such that:

$$\Sigma \models \exists \bar{x} \, \exists^{=1} y \left( \left( \bigwedge_{i=1}^{n} y \neq x_i \right) \wedge \varphi(\bar{x}, y) \right).$$

"There is no formula in $F$ that <u>uniformly witnesses</u> non-trivial definable closure in all models of $\Sigma$."

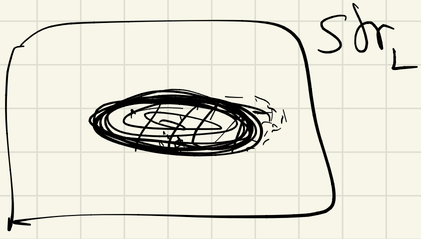
So far: We've talked about

$S_\infty$-invariant measures.

A special subclass of such measures: the ergodic ones

An invariant measure $\mu$ on $Str_L$ is ergodic if for any Borel subset $X$ of $Str_L$ that is invariant under the Logic action up to a set of measure 0, i.e. $\mu(X \Delta g(X))=0$ for all $g \in S_\infty$, we have:

$\mu(X)=0$ or $\mu(X)=1$.



FACT: The set of invariant measures on $Str_L$ is a convex set. The extreme points of this set are precisely the ergodic invariant measures; any invariant measure is a mixture of extreme ones.

$\therefore$ Wlg, we may consider only ergodic invariant measures on $Str_L$.

(one direction of Lopez-Escobar Theorem)

<u>Observe</u>: Extents of sentences are Borel and invariant under the logic action. For any sentence $\varphi$ + ergodic $\mu$, we have $\mu(\llbracket \varphi \rrbracket) = 0$ or $1$.

We've seen: extents of $\mathcal{L}_{\omega_1,\omega}$-sentences are given measure 0 or 1 by an $\underbrace{\text{ergodic invariant measure}}$.

<u>Defn</u>: $Th(\mu) = \{ \mathcal{L}_{\omega_1,\omega}$ sentences $\phi$ :
$$\mu([\![ \phi ]\!]) = 1 \}.$$

<u>Note</u>: $\mu$ ergodic invariant. Then:

(a) $Th(\mu)$ is complete.

by ergodicity.

(b) countably satisfiable

Any countable subset of $Th(\mu)$ has a model.

because countable intersection of measure 1 sets is measure 1.

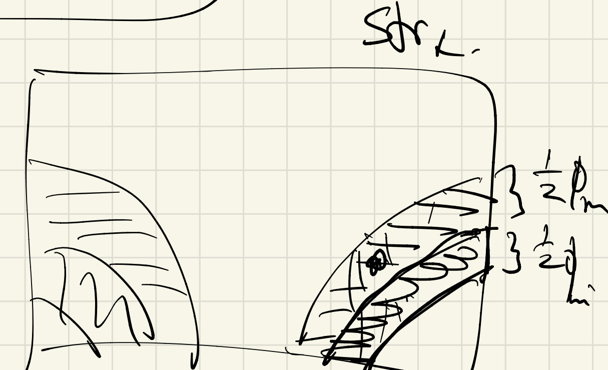
So:

One could propose "ergodic invariant measures as a notion of "probabilistic structure"."

FACT (Ackerman-Freer-Kruckman-P.)

If $\mathcal{M}$ is properly ergodic then $Th(\mathcal{M})$ has no models (of any cardinality).

eg. Properly ergodic: Kaleidoscope random graph: $\omega$-many "edge colours" any of which can hold between 2 vertices with probability $\frac{1}{2}$ is properly ergodic

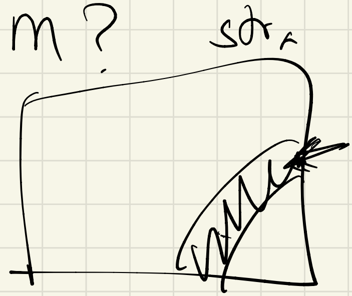



Because of countable consistency: An ergodic invariant measure assigns (a) measure 1 to an orbit OR (b) measure 1 to a continuum of orbits while assigning measure 0 to any particular orbit.

$\mathcal{M}$ ergodic invariant

In case (b), we say $\mathcal{M}$ is "properly ergodic".

**Q:** Given $M \in \mathrm{Str}_L$, how many ergodic invariant measures are concentrated on orbit of $M$?




e.g. $M = \mathcal{R}$, we know answer is continuum bec. each $G(\mathbb{N}, p)$, $0<p<1$ is an ergodic invariant measure.

e.g. $M = \mathbb{Q}$, we know answer is 1 (Glasner–Weiss).

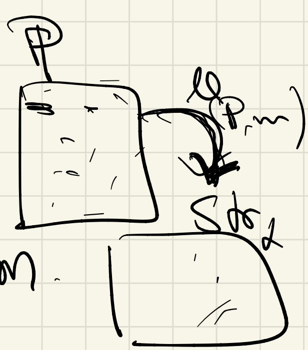
FACT: $\mu_{(\mathbb{P},m)}$ from construction yesterday is ergodic.

Ans: (Ackerman–Freer–Kwiatkowska–P.)

0, 1 or continuum.

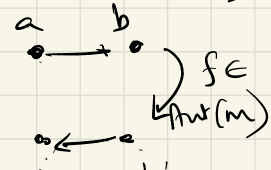
1 → we can describe these $M$.

0 → AFP16, when $M$ has non-trivial dcl.



The case "1" occurs precisely when $M$ is highly homogeneous.

**Defn** (Peter Cameron): $M \in Str_L$ is *highly homogeneous* when for each $k < \omega$ and every pair of $k$-element sets $X, Y \subseteq M$, there is $f \in Aut(M)$ s.t. $Y = \{f(x) : x \in X\}$.




**Thm** (AFKP): No. of ergodic inv. measures on orbit of $M$ is $1$ iff $M$ is highly homogeneous.

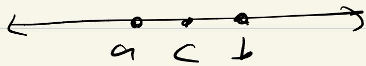
**FACT** (Peter Cameron): Any highly homogeneous structure is *interdefinable* (has same definable sets) with one of the 5 reducts of $\mathbb{Q}$.

What are the 5 reducts of $\mathbb{Q}$?

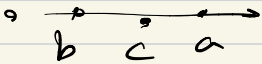
① Pure set.

② $(\mathbb{Q}, \leq)$.

③ ternary "betweenness" relation $B$



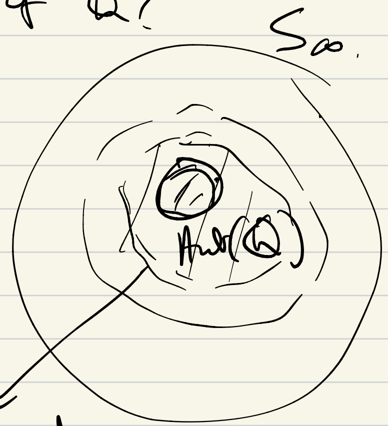

$B(a,c,b) \Leftrightarrow$ $a<c$ and $c<b$. or $b<c$ and $c<a$




④ ternary circular order:

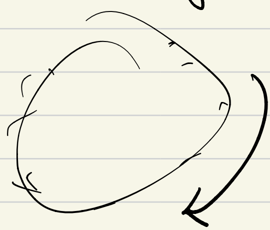
Wrap ③ around a circle

⑤ quaternary separation relation:




closed subgroup of $S_\infty \supseteq \mathrm{Aut}(\mathbb{Q})$ will be the Automorphism group of a structure.

This structure is called a reduct of $\mathbb{Q}$.



ignore clockwise vs. counter-clockwise in ④

This leads to the obvious more general question: How many "ergodic models" does a given extent have?

ergodic invariant measure concentrated on the extent

Ans: (essentially Ackerman–Freer–Kwiatkowska–Kruckman–P.).

Any $\underline{n \leq \omega}$ or continuum.

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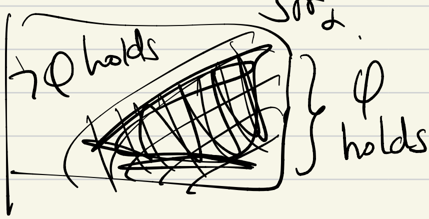
Recall: $\varphi$ an $L_{\omega_1,\omega}$-sentence, extent of $\varphi$ is $[\![\varphi]\!] =$

$\{M \in Str_L : M \models \varphi\}$



Notice: There is an $L_{\omega_1\omega}$-sentence defining high homogeneity:

$$\Psi : \bigwedge_{n<\omega} \Big( \forall x_0 \ldots x_{n-1}, y_0 \ldots y_{n-1} \Big( x_i \text{ distinct} \wedge y_i \text{ distinct}$$

$$\rightarrow \bigvee_{\sigma \in S_n} \bigwedge_{\psi \in L_{\omega\omega}(L)} \psi(x_0 \ldots x_{n-1}) \leftrightarrow \psi(y_{\sigma(0)}, \ldots, y_{\sigma(n-1)}) \Big)\Big)$$

NOTE: high homogeneity $\Rightarrow$ $\aleph_0$-categorical.

$\Theta$ $L_{\omega_1,\omega}$-sentence

$\Theta$ fails syntactic trivial dcl.

Ans: $0$

$\Theta$ has syntactic trivial dcl.

Ans: $>0$

$\Theta \wedge \neg \Psi$ has syntactic trivial dcl.

Ans: $2^{\aleph_0}$.

$\Theta \wedge \neg \Psi$ fails syntactic trivial dcl.

$\Theta \wedge \Psi$ is Scott sentence.

Ans: $1$

$\Theta \wedge \Psi$ not a Scott sentence.

$1 < n \leq \omega$ ??

$2^{\aleph_0}$ ??

$\aleph_1$ ??

$\Theta \wedge \Psi$ not a Scott sentence

for any $n \leq \omega$, we can find examples.

Similarly can find examples with $2^{\aleph_0}$

What about $\aleph_1$?

**Proposition:** Suppose $\Theta$ is an $L_{\omega_1\omega}$ sentence with $< 2^{\aleph_0}$-many highly homogeneous models (up to isomorphism). Then $\Theta$ has only countably many highly homogeneous models.

__Pf__ Let $\equiv$ be the equiv. reln. on $Str_L$ such that $\mathcal{M} \equiv \mathcal{N}$ iff:

$\hookrightarrow$ $\left.\begin{array}{l} \mathcal{M} \cong \mathcal{N} \models \Theta \wedge \Psi \\ \text{\underline{OR}} \quad \mathcal{M}, \mathcal{N} \models \neg(\Theta \wedge \Psi) \end{array}\right\}$

Then $\equiv$ is ~~an~~ a Borel equiv. relation on $Str_L$. Silver's Dichotomy says: $Str_L/\equiv$ is countable or size $2^{\aleph_0}$.