PROBABILISTIC CONSTRUCTIONS IN MODEL THEORY

DAY 3

Yesterday, we saw:

$\mathcal{R}$, $\mathcal{B}$ have nice constructions:

→ Fraïssé limits

A → "coin-flipping" random construction,

B → almost sure theory from a 0-1 law.

Henson graph $\mathcal{H}$: Is a Fraïssé limit

but A-type construction ✗

B-type construction ✗

Can we get a random construction of $\mathcal{H}$ that has the nice symmetry property of A: invariance under reordering of underlying set.

A random structure whose distribution is invariant under permutations of the underlying set is called <u>exchangeable</u>.

<u>Q</u>: Does H have an exchangeable construction?

<u>Ans</u>: Yes! Petrov-Vershik (2010)

Formal setting in which to ask this question:

$\mathcal{L}$: countable language.

$\mathrm{Str}_{\mathcal{L}}$: space of all $\mathcal{L}$-structures with underlying set $\mathbb{N}$.

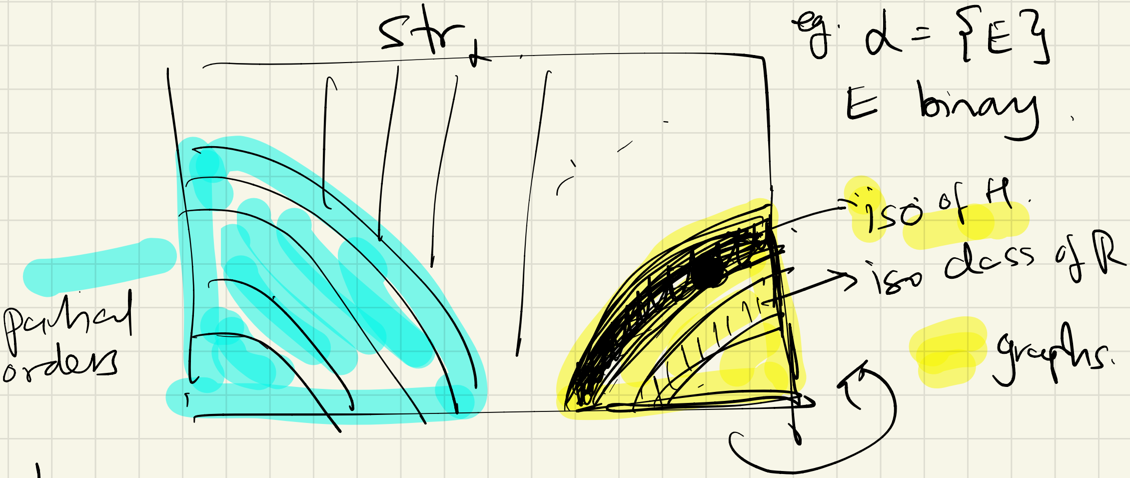
Str_L is a measure space in a standard way: by equipping it with the Borel σ-algebra generated by subbasic open sets of the form:

$$\{M \in Str_L : M \models E(\bar{a})\}$$

and $\{M \in Str_L : M \models \neg E(\bar{a})\}$

for each relation symbol E in L and tuple $\bar{a} \in \mathbb{N}$ with $|\bar{a}| = \text{arity}(E)$.

(Similarly for function + constant symbols).



Logic Action: The group S_∞ of permutations on $\mathbb{N}$ acts on Str_α via so-called <u>logic action</u>: For $g \in S_\infty$ & $m \in \mathrm{Str}_\alpha$, $g \cdot m \in \mathrm{Str}_\alpha$ is a relabeling of m by g.

Notice: Orbits in Str_α under logic action are precisely isomorphism classes of α-structures.

Invariant probability measures on Str_L

A prob measure μ on Str_L is (S_∞)-invariant if the logic action doesn't change the measure of a Borel set in Str_L.

i.e for any Borel $X \subseteq \mathrm{Str}_L$, $g \in S_\infty$,

$$\mu(X) = \mu(g \cdot X).$$

NOTE: Orbits in Str_L are Borel.

NOTE: A random structure in Str_L is exchangeable precisely when its distribution is invariant under the logic action.

Defn.:
A prob. measure μ on Str_L is
concentrated on Borel X in Str_L
when $\mu(X) = 1$. In this case, we
say X "admits" an invariant
measure.

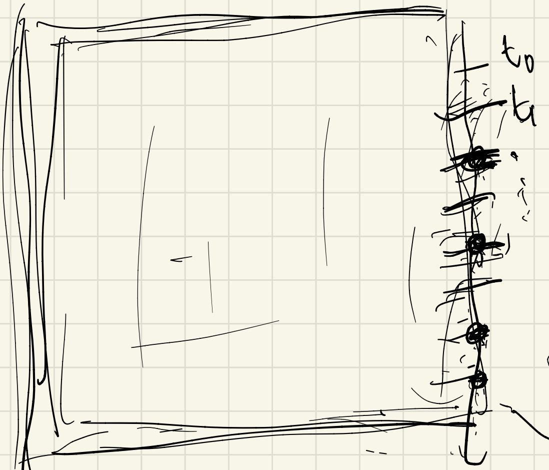
This raises a general question:
$\leadsto$ Which countable structures
arise almost surely from exchangeable
constructions?

$\iff$ Given $M \in Str_L$, does M
admit an invariant measure?

How do Petrov-Vershik show existence of an invariant measure on Str_L concentrated on isomorphism class of $\mathcal{H}$?

Rough description, adapted to methods of Ackerman-Freer-P:

$L = \{E\}$
E binary.



$\mathbb{P}$

Borel graph on $\mathbb{R}$.

$\mathcal{J}_{\mathbb{P}} : \mathbb{R}^{\omega} \longrightarrow Str_L$,

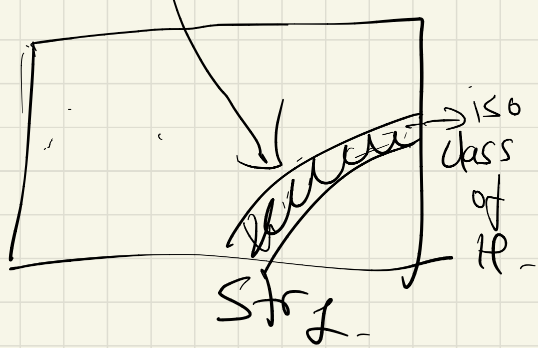
$\bar{a} \in \mathbb{N}$.

$E(a_0 \ldots a_{k-1})$ iff $E(t_{a_0}, \ldots, t_{a_{k-1}})$

Define:
$\mu_{(\mathbb{P},m)} = m^{\infty} \circ \mathcal{J}_{\mathbb{P}}^{-1}$

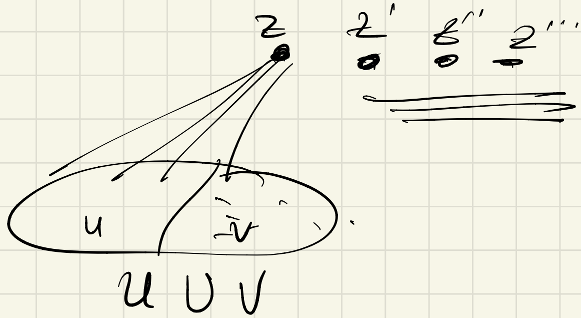
Build $\mathbb{P} \models Th(\mathcal{H})$ with special properties.

m a nice measure on $\mathbb{R}$ (continuous, non-degenerate)



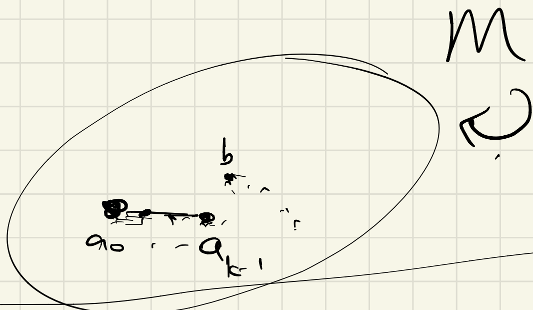
How can we build a $\mathbb{P}$ with the required special properties?

<u>Recall</u>: $\mathbb{P}$ has free amalgamation.
This allows us to "clone" witnesses to extension axioms as much as we wish. (in fact,



strong amalgamation suffices)

Defn: For $\mathcal{M} \in \mathrm{Str}_L$ + finite tuple $\bar{a} \in \mathbb{N}$, the <u>group-theoretic definable closure</u> of $\bar{a}$ in $\mathcal{M}$, denoted $\mathrm{dcl}_{\mathcal{M}}(\bar{a})$, is the collection of $b \in \mathcal{M}$ that are fixed by all automorphisms of $\mathcal{M}$ fixing $\bar{a}$ pointwise.



Theorem: (Ackerman-Freer-P. 2016):

TFAE:

① $\mathcal{M}$ admits an invariant measure.

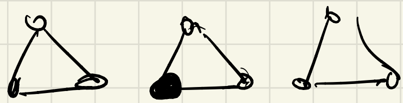
② $\mathcal{M}$ has trivial group-theoretic definable closure, i.e. $\mathrm{dcl}_{\mathcal{M}}(\bar{a}) = \bar{a}$ for any $\bar{a} \in \mathcal{M}$.

FACT: For $\mathcal{L}$ finite relational language + $\mathcal{M}$ an ultrahomogeneous $\mathcal{L}$-structure, trivial group theoretic definable closure = strong amalgamation.

Corollary: The following have exchangeable constructions

✓ $\mathcal{R}$, $\mathcal{H}_n$, $\mathcal{B}$ $\longrightarrow$ known

✓ $\mathcal{P}$ $\rightsquigarrow$ not previously known.

✗  . . . . . easy to see even without main theorem.

Recap: Wanted probabilistic constructions of

$\checkmark \mathcal{R} \longrightarrow$ independence

$-\mathcal{H} \longrightarrow$ exchangeable

$\checkmark \mathbb{Q} \longrightarrow$ exchangeable

$-\mathbb{P} \longrightarrow$ exchangeable

$\checkmark \mathbb{B} \longrightarrow$ independence

Thm: invariant measure (M has exchgble construction) $\iff$ trivial definable closure.

In case of Fraïssé limits in finite relational language:

trivial definable closure

$\iff$ strong amalgamation.

(Non)- Examples of countable infinite structures with exchangeable constructions in a this theorem:

<u>Non-Examples</u>:

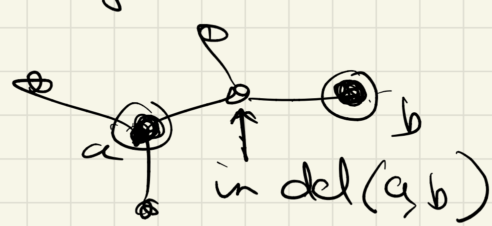
→ If $\mathcal{L}$ has a constant symbol, no exchangeable construction.

→ If $\mathcal{L}$ has function symbols, these must all be interpreted as "choice functions" in M in order for M to have an exchangeable construction.

" $f(\bar{a}) \in \bar{a}$ "

→ No tree has an exchangeable construction

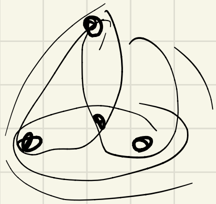
eg.



Examples: H_n: Petrov-Vershik

$\mathbb{P}$ new

generic tetrahedron-free 3-uniform hypergraph: free amalgamation.



tetrahedron = complete 3-uniform hypergraph on 4 vertices.

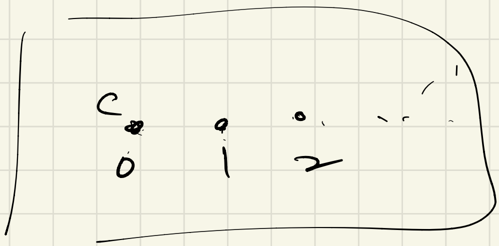
eg. infinitely many infinite equivalence classes with non-$\aleph_0$-categorical structure on the quotient will be a non-$\aleph_0$-categorical structure with trivial dcl.

eg. The universal existentially complete graph omitting $\{C_3, C_5, \dots, C_{27}\} = \mathcal{S}$ as weak substructures.

Cherlin Shelah Shi: This is universal for all $\mathcal{S}$-free graphs, trivial dcl, $\aleph_0$-cat, not ultrahomog.

Easy to see that non-trivial definable closure $\Longrightarrow$ no exchangeable construction.

<u>Example</u>: $\mathcal{L}$ has a constant symbol c



Then prob. that c holds of any given $i \in \mathbb{N}$ is the same as for any other element of $\mathbb{N}$ — but these are disjoint events

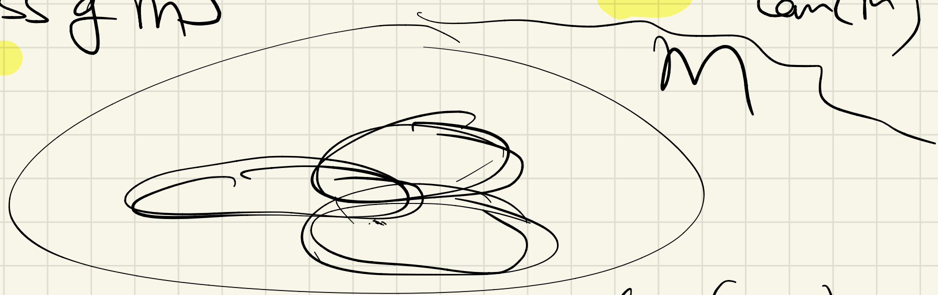
So by countable additivity, contradiction

Note: Main Theorem applies to arbitrary countable infinite structures, not necessarily Fraïssé limits that are axiomatised by "one-point extension axioms"

How do we adapt the proof for H (Petrov–Vershik) and other Fraïssé limits to this more general context?

We "Morleyise" and move to a so-called canonical structure in a canonical language which has the same definable sets as original structure.

$\mathcal{M}$	Can($\mathcal{M}$), Canonical structure for $\mathcal{M}$
=	=
trivial dcl $\longleftrightarrow$	trivial dcl
invariant measure on iso class of $\mathcal{M}$ $\longleftrightarrow$	invariant meas. on iso class of Can($\mathcal{M}$)



Can($\mathcal{M}$)

$\mathcal{M}$ may not be axiomatised by 1-point extension axioms

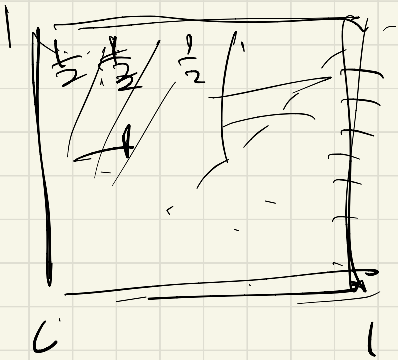
BUT

Can($\mathcal{M}$) is ultrahomogeneous & axiomatised by 1-point extension axioms in $\underbrace{\mathcal{L}_{\omega_1,\omega}}_{\text{not first-order.}}$

Connection with Graphons:

What is a graphon?

A symmetric measurable function $f : [0,1]^2 \longrightarrow [0,1]$



Can think of it as an edge-weighted graph on $[0,1]^2$

"Erdős-Rényi graphon"

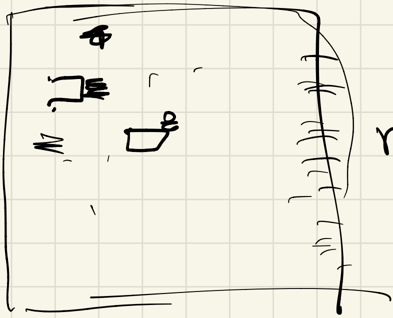
eg. f: constant function $\frac{1}{2}$ for (x,y) s.t. $x \neq y$, 0 otherwise

Lovasz, Szegedy, many others have developed theory of graphons recently (+ Vershik similar work). Notice that sampling a graphon produces an invariant measure on Str_L as before.

where $L = \{E\}$, E binary

When the target $\mathcal{M}$ is a graph, the Borel structures $\mathbb{P}$ that we build are precisely graphons, but with an additional feature: they are maps to $\{0,1\}$, <u>not</u> $[0,1]$.

We get not edge-weighted graph but an actual graph as $\mathbb{P}$.



$\mathbb{P}$.

These are called <u>random-free</u> graphons.

Petrov–Vershik's continuum-sized graphs are random-free graphons.

In fact, Petrov–Vershik have a new exchangeable construction of Rado graph via this method of sampling a random-free graphon

Corollary of Main Theorem (AFP):

If a graph (any countable infinite structure) has an exchangeable construction, then it has one that comes from sampling a random-free graphon.

Aldous-Hoover-Kallenberg Theorem

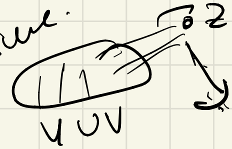
(+ translation by Ackerman for all countable languages): Representation Theorem for exchangeable structures that say <u>any</u> exchangeable structure arises as a mixture of generalisations of such sampling procedures.

Construction of $\mathbb{P}$:

See "Invariant measures concentrated on countable structures"

Ackerman Freer-Patel.

→ Build $\mathbb{P} \models Th(\mathcal{H})$ with property that every extension axiom in $Th(\mathcal{H})$ is realised on an $\underbrace{\text{interval of } \mathbb{R}}_{\text{set of positive } m\text{-measure.}}$



→ Let m be a non-degenerate continuous measure on $\mathbb{R}$.

→ Sample an m-i.i.d. sequence from $\mathbb{R}$.

→ Let $\mu_{(\mathbb{P},m)}$ be the "pushforward" measure on Str_L. It is the distribution of a random structure in Str_L.

→ If we can build such $\mathbb{P}$, $\mu_{(\mathbb{P},m)}$ will be an invariant measure concentrated on $\mathcal{H}$.