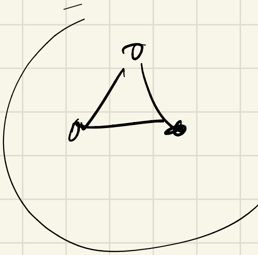
PROBABISTIC CONSTRUCTIONS IN MODEL THEORY

DAY 2



Yesterday: Examples of Fraïssé limits.

Structure	Notation	Age
Rado graph	$\mathcal{R}$.	all finite graphs.
Henson graph	$\mathcal{H}$	all triangle-free finite graphs
"generic partial order"	$\mathcal{P}$	all finite partial orders
random bipartite graphs	$\mathcal{B}$	all finite bipartite graphs in language $\{E, B, R\}$

Q Which graphs are Fraïssé limits?

Lachlan-Woodrow classification (1980):
The following are the ultrahomogeneous countable infinite graphs (in language with just the edge relation).

$\hookrightarrow$ Rado graph

$\hookrightarrow$ H_n, for $n \geq 3$ $\qquad H = H_3$.
"Henson graphs"
or complements.

K_n = complete graph on n vertices

FACT: For fixed $n \geq 3$, finite K_n-free graphs form amalg. class, hence have Fraïssé limit, denoted H_n.

$\hookrightarrow$ $n \cdot K_\omega$, for any $n \leq \omega$. n-many copies of K_ω.
or complement.

$\hookrightarrow$ $\omega \cdot K_n$ for each $n < \omega$. ω-many copies of K_n.
or complements.

NOTE: ~~$\mathcal{R}$~~ $\mathcal{R}$ is in some sense a "limiting structure" for all finite graphs.

↪ universal.

↪ Fraïssé limits are "built up" from finite pieces.

Q Is there some random graph that, in a similar way, captures ~~more~~ the structure of all finite graphs?

First Order Labelled 0-1 Laws

For today: $\mathcal{L}$ finite relational language

Let C be a class of finite $\mathcal{L}$-structures closed under isomorphism.

For $n < \omega$, let C_n be the set of elements of C with universe $\{0, \dots, n-1\}$

C_n = all members of C with universe $\{0, \dots, n-1\}$

We'll distinguish isomorphic but unequal members of C, "labelled"

e.g. for graphs: $0 - 1 - 2 \neq 0 - 2 - 1$

We'll think of C_n as a finite probability space, where each member is weighted equally.

Want to consider, for a given first order $\mathcal{L}$-sentence ϕ, the probability that ϕ holds in C_n.

Defn: Given first-order $\mathcal{L}$-sentence φ, $n<\omega$, let

$$\mu_n(\varphi) = \frac{|\{\mathcal{M} \in \mathcal{C}_n : \mathcal{M} \models \varphi\}|}{|\mathcal{C}_n|}$$

assume all $\mathcal{C}_n$ non-empty.

The (labelled) asymptotic probability of φ with respect to $\mathcal{C}$ is:

$$\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$$

when this limit exists; otherwise it is undefined.

If this limit exists, μ is a finitely additive prob measure on $\mathcal{L}$-sentences.

Digression: We are looking at first-order sentences, where

C_n has uniform prob. measure.

Could change the logic

Could change the measure on C_n.

Won't do this today.

We'll consider different C.

Defn: $\mathcal{C}$ has a first-order labelled 0-1 law if, for every first-order $\mathcal{L}$-sentence φ, $\mu(\varphi)$ exists & equals either 0 or 1.

Further, if $\mu(\varphi) = 1$, we say φ holds (asymptotically) almost surely on $\mathcal{C}$, or "almost every member of $\mathcal{C}$ satisfies φ". If $\mu(\varphi) = 0$, we say φ holds (asymptotically) almost never on $\mathcal{C}$

NOTE: If $\mathcal{C}$ has a first-order 0-1 law, this tells us something about the expressive power of $L_{\omega\omega}$ about $\mathcal{C}$.

first-order.

Defn: The set of all first-order $\mathcal{L}$-sentences ϕ for which $\mu(\varphi)=1$ is the first order almost sure theory for $\mathcal{C}_0$, which we'll denoted

T^{as}.

FACTS:

EX (1) For any φ, $\mu(\varphi)=1$ iff $\mu(\neg\varphi)=0$

EX (2) For any φ + ψ, $\mu(\varphi)=\mu(\psi)=1$

then $\mu(\varphi\wedge\psi)=1$.

(3) T^{as} is deductively closed

For any ϕ, if $T^{as}\vdash\varphi$, then by Compactness, there are $\psi_0\ldots\psi_{k-1}\in T^{as}$ s.t. $\{\varphi_0\ldots\varphi_{k-1}\}\vdash\phi$. Then

EX apply (2) to get that $\phi\in T^{as}$

(4) T^{as} is consistent.

Use that $\mathcal{M}(\exists x(x \neq x)) = 0$

(5) T^{as} does not have finite models.

★★★ (6) T^{as} has the <u>finite model property</u>.

<u>Defn</u>: A theory T has the <u>finite model property</u> if every sentence in the theory has a finite model.

A model $M \models T$ has the <u>finite submodel property</u> for T if every sentence of T holds in some finite substructure of M.

(7) $\mathcal{C}$ has a first-order labelled 0-1 law iff T^{as} is complete.

(Immediate from (1) & (3)).

Goal: (a) Given some $\mathcal{C}$, does $\mathcal{C}$ have a first-order labelled 0-1 law?

(b) If yes, what is T^{as}?

(c) If $\mathcal{C}$ is an amalgamation class, is $T^{as} = \mathrm{Th}(\text{Fraïssé limit of } \mathcal{C})$?

(d) If T is a complete theory, is $T = T^{as}$ for some $\mathcal{C}$?

Back to our set of favourite examples.

structure	class C
$\to \mathcal{R}$	$\mathrm{age}(\mathcal{R})$ = all finite graphs.
? $\mathcal{H}$	$\mathrm{age}(\mathcal{H})$ = all Δ-free finite graphs
$\times$ $\mathbb{Q}$	$\mathrm{age}(\mathbb{Q})$ = all finite linear orders. $\mathrm{Th}(\mathbb{Q})$ can't be a T^{as} because density has no finite models.
? $\mathbb{P}$	$\mathrm{age}(\mathbb{P})$ = all finite partial orders.
$\to \mathcal{B}$	$\mathrm{age}(\mathcal{B})$ = all finite b.p. graphs $\mathcal{L} = \{E, R, B\}$.

Which of these has a theory that equals T^{as} for T^{as} the almost sure theory of some C?

For now: let's look at some classes C, & discuss if they have 0-1 laws.

$\boxed{\mathcal{L}: \text{finite relational}}$

Example (A) C = all finite $\mathcal{L}$-structures

Ans: Yes, C has a 0-1 law.

Proved by: Glebskii, Kogan, Liagonkii, Talanov (1969), & independently Fagin (1976).

(B) C = class of all finite graphs.

Ans: Yes, C has a 0-1 law.

Almost sure theory of C = Th(Rado graph).

Pf: Similar to Fagin argument for (A) & to $G(\mathbb{N}, \frac{1}{2})$ Erdős-Rényi argument.

To show C = all finite graphs has 0-1 law with $T^{as} = Th(\mathcal{R})$, it suffices to show that axioms characterising $\mathcal{R}$ have prob. 1 w.r.t. C.

Axioms for $\mathcal{R}$:

- irreflexivity
  $\forall x \, \neg(x E x)$ ✓ $\mu = 1$ here.
- symmetry:
  $\forall x \forall y (x E y \to y E x)$ ✓
- Convert property $(\cancel{*})$ into first-order sentence:

For $k, l \leq \omega$

$$\psi_{k,l} : \forall x_1 \ldots x_k \, \forall y_1 \ldots y_l \, (x_i, y_j \text{ distinct} \to \exists z \Big(\bigwedge_{i<k} z \neq x_i \wedge \bigwedge_{j<l} z \neq y_j \wedge \Big(\bigwedge_{i<k} z E x_i \Big) \wedge \Big(\bigwedge_{j<l} \neg z E y_j \Big) \Big).$$

Notice: We had uniform measure on C_n

$$C_n = \{ M \text{ in } C : \text{universe of } M \text{ is } \{0, \dots, n-1\} \}.$$

This is saying, essentially, that the probability of an edge between two particular vertices $i, j < n$ is $\frac{1}{2}$.

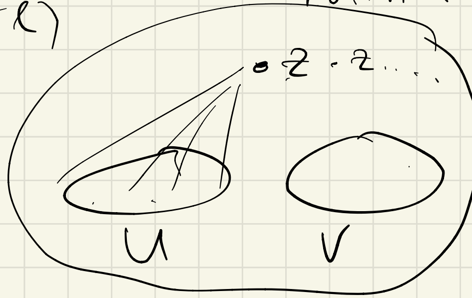
Consider $\varphi_{k,\ell}$, where $k, \ell < n$
$(k+\ell < n)$

Look at $\mu_n(\varphi_{k,\ell})$ & let $n \to \infty$.

For given U, V where $|U| = k$, $|V| = \ell$,

$$Pr(U \,\&\, V \text{ don't satisfy } \varphi_{k,\ell})_{\{0..n-1\}}$$

$$= \left(1 - \frac{1}{2^{k+\ell}}\right)^{n-(k+\ell)}$$



There are $\binom{n}{k}\binom{n-k}{\ell}$ such U & V.

(next page)

Prob that $\psi_{k,\ell}$ fails on C_n

$$\leq \underbrace{\binom{n}{k}\binom{n-k}{\ell}}\ \underbrace{\left[1-\frac{1}{2^{k+\ell}}\right]}_{<1}^{n-k-\ell}$$

k, l constant

$\xrightarrow{n\to\infty} 0.$

ie $\mu(\neg\psi_{k,\ell}) = 0$

$\mu(\psi_{k,\ell}) = 1.$

$\psi_{k,\ell} \in T^{as}$ for each k, ℓ.

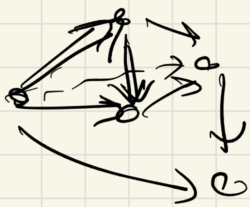
By properties mentioned earlier ((4)?), for any $\phi \in Th(\mathcal{R})$, $\phi \in T^{as}$ & vice versa.

For finite graphs: 0-1 law exists.

Th (Fraïssé limit) = T^{as}.

$\mathcal{R}$ is the unique (up to isomorphism) model of this theory.

<u>Notice</u>: A similar argument will work for C = all finite tournaments



$\mathcal{L} = \{E\}$, E binary relation

For any a, b, $a \neq b$, aEb <u>OR</u> bEa but not both or neither

+ irreflexive.

Also for C = all 3-uniform hypergraphs.

$\mathcal{L} = \{F\}$, F ternary.

Relation is symmetric, holds of distinct elements.

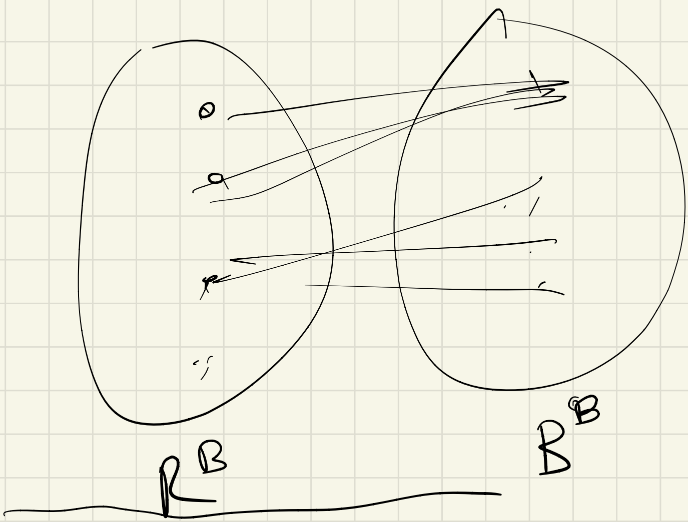
Q Will this type of argument hold for $\mathcal{B}$?

$\mathcal{L} = \{E, R, B\}$

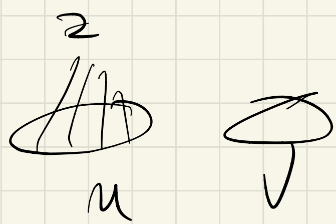
E: binary, irreflexive, symmetric

R, B: unary, every vertex is an R or B but not both

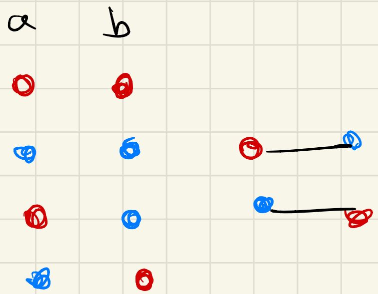
M: Fraïssé limit $\mathcal{B}$:



For all $U, V \subseteq R^{\mathcal{B}}$, $\exists z \in B^{\mathcal{B}}$



& similar for all $U', V' \subseteq B^{\mathcal{B}}$



0-1 law with $\mathrm{Th}(\text{Fraïssé limit}) = T^{as}$.

Our list:

$R \hookrightarrow$ age has 0-1 law, $T^{as} = Th(R)$.

?? H

$\mathbb{Q} \hookrightarrow$ age has 0-1 law, $T^{as} \neq Th(\mathbb{Q})$.

?? $\mathbb{P}$

$\mathbb{B} \hookrightarrow$ age has 0-1 law, $T^{as} = Th(\mathbb{B})$.

Let's consider $\mathcal{C}$ = all finite Δ-free graphs

[In general, could consider K_n-free graphs, fixed $n \geq 3$]

<u>Thm</u> (Erdös-Kleitman-Rothschild (1973), Kolaitis-Prömel-Rothschild 1987)

Yes, $\mathcal{C}$ (= class of finite Δ-free graphs) has a 0-1 law.

<u>Q</u>: Is $T^{as} = Th(\mathcal{H})$?? No!!

[In fact, similar for any $\mathcal{C}$ = finite K_n-free graphs]

What is T^{as} for finite $\triangle$-free graphs?

EKR: Almost every $\triangle$-free graph is bipartite.

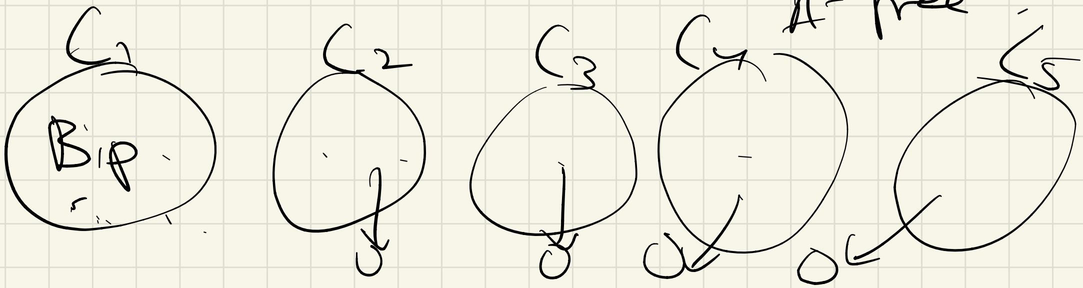
KPR: Almost every K_n-free graph is $(n-1)$-partite.

Formally:

$$\lim_{n\to\infty} \frac{|\{\text{bipartite graphs in } C_n\}|}{|C_n|} \to 1.$$

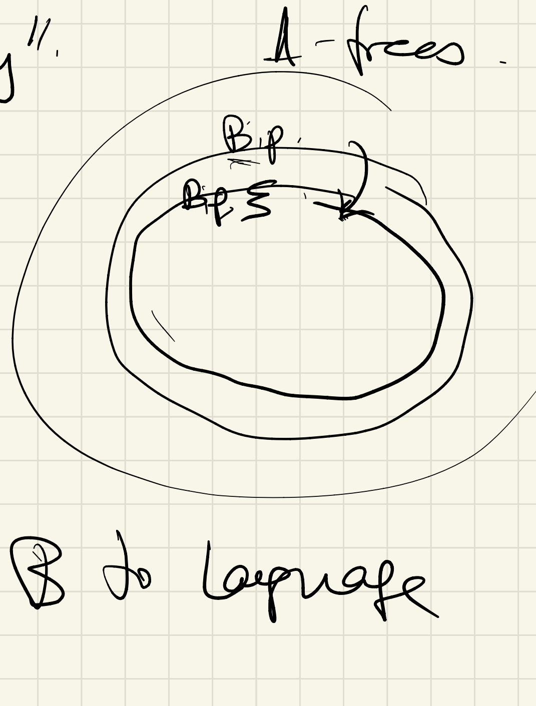
"EKR Method"

C = all finite $\triangle$-free



KPR showed:

Almost every finite Δ-free graph has a first-order definable property Σ that implies unique "bipartition-ability".



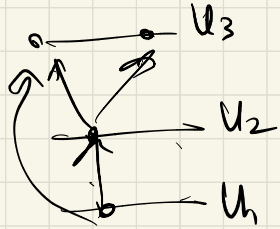
Thm (KPR 1987): For C = all finite Δ-free graphs,

T^{as} = reduct of $\mathbb{B}$ to language $\{E\}$.

Further, T^{as} for C = all finite K_n-free graphs is reduct, to $\mathcal{L} = \{E\}$, of generic $(n-1)$-partite graph in $\mathcal{L} = \{E, \overbrace{U_0, \ldots, U_{n-2}}^{\text{unary}}\}$

Also, EKR + Compton:

$\mathcal{C}$ = all finite partial orders has a 0-1 law. They show: almost every finite partial order has height 3.



Reduct of a Fraïssé limit in language $\{E, \underbrace{U_1, U_2, U_3}_{\text{give 3 levels}}\}$.

Q We know $Th(\mathcal{H}) \models T^{as}$ for C = finite Δ-free graphs.

Is $Th(\mathcal{H})$ the limiting theory, in any sense, of finitary random processes?

BIG OPEN Q: Does $\mathcal{H}$ have the finite model property?

→ Greg Cherlin

Axioms for $\mathcal{H}$:

irreflexive

symmetric

no Δs

+ "for all U, V finite disjoint, U an independent set, $\exists z \notin U \cup V$ st zEx for all $x \in U$, & $\neg zEy$ all $y \in V$"