## Scott complexity of countable structures

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Can we measure this complexity? How?

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Three tools used to measure complexity:

- Back-and-forth relations for orbit complexity.
- Borel/Baire complexity for identification and isomorphism complexity.
- Infinitary formulas for orbit and identification complexity.

Theorem: [M. 15] Let  $\alpha$  be a successor ordinal and A a countable structure. The following are equivalent:

- All automorphism orbits in  $\mathcal{A}$  are  $\Sigma_{\alpha}^{in}$ -definable.
- The set of representations of  $\mathcal{A}$  is  $\Pi_{\alpha+1}$  in the Borel hierarchy.
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#### Definition: Let Rank(A) be the least such $\alpha$ .

Other definitions of rank had been proposed: [Scott 65][Sacks 07][Ash-Knight 00]... However, this is the first equivalence theorem.



#### Technique 1: Back-and-forth relations for orbit complexity.

Ash-Knight's version, in the spirit of Scott's original definition.

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•  $(a_1, ..., a_k) \cong_{\mathbb{Q}} (b_1, ..., b_k).$ 

Thus,  $\bar{a} \equiv_0 \bar{b} \iff \bar{a} \cong \bar{b}$ , and  $akRank(\mathbb{Q}; <) = 0$ .

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# Consider the order $(\mathbb{Z}; <)$ : • $(a_1, ..., a_k) \equiv_0 (b_1, ..., b_k) \iff (\forall i, j < k) \quad a_i < a_j \leftrightarrow b_i < b_j.$ • $(\Rightarrow) \quad (a_1, ..., a_k) \cong_{\mathbb{Z}} (b_1, ..., b_k).$ • $(a_1, a_2) \leq_1 (b_1, b_2) \iff (a_1, a_2) \equiv_0 (b_1, b_2) \text{ and } |a_2 - a_1| \geq |b_2 - b_1|.$

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•  $(a_1, ..., a_k) \equiv_0 (b_1, ..., b_k) \iff (\forall i, j < k) \quad a_i < a_j \leftrightarrow b_i < b_j.$   
•  $(\Rightarrow) (a_1, ..., a_k) \cong_{\mathbb{Z}} (b_1, ..., b_k).$   
•  $(a_1, a_2) \leq_1 (b_1, b_2) \iff (a_1, a_2) \equiv_0 (b_1, b_2) \text{ and } |a_2 - a_1| \geq |b_2 - b_1|.$   
•  $(\Rightarrow) (a_1, a_2) \cong_{\mathbb{Z}} (b_1, b_2).$   
 $\bar{a} \cong \bar{b} \iff \bar{a} \leq_1 \bar{b} \& \bar{b} \leq_1 \bar{a}.$ 

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•  $(a_1, a_2) \leq_1 (b_1, b_2) \iff (a_1, a_2) \equiv_0 (b_1, b_2) \text{ and } |a_2 - a_1| \geq |b_2 - b_1|.$   
•  $(\Rightarrow) (a_1, a_2) \cong_{\mathbb{Z}} (b_1, b_2).$   
 $\bar{a} \cong \bar{b} \iff \bar{a} <_1 \bar{b} \& \bar{b} <_1 \bar{a}.$  Therefore  $akRank(\mathbb{Z}; <) = 2.$ 

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Transfinite recursion:  $(\forall H: \mathcal{X}^{ordinals} \to \mathcal{X}) (\exists F: ordinals \to \mathcal{X})$ 

 $F(\alpha) = H(F \upharpoonright \{\beta : \beta < \alpha\})$ 

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(Do not read:) In reality we use the back-and-forth rank:  $bfRank(\mathcal{A}) = \text{least } \alpha \quad \forall \bar{a} \in A^{<\mathbb{N}} \exists \bar{a}' \in A^{<\mathbb{N}} \forall \bar{b}\bar{b}' \in A^{<\mathbb{N}}, \quad \bar{a}\bar{a}' \leq_{\alpha} \bar{b}\bar{b}' \Rightarrow \bar{a} \cong_{\mathcal{A}} \bar{b}$ 

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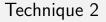
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Lemma: [M. 15]  $akRank(A) \leq bfRank(A) \leq akRank(A) + 1$ .



# Technique 2: Borel/Baire complexity for identification and isomorphism complexity.

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Definition: Let  $\mathcal{X}_{\tau}$  be the set of structures on vocabulary  $\tau$  and domain  $\mathbb{N}$ .

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Equip  $\mathcal{X}_{\tau}$  with the topology from  $\mathbb{N}^{\mathbb{N}}$  given by the power of the discrete topology.

## The space of structures $\mathcal{X}_{ au}$

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We can measure identification complexity of A in terms of the Borel complexity of Copies(A).

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A  $\Sigma_n$  subset of  $\mathcal{X}_{\tau}$  (or of  $\mathbb{N}^{\mathbb{N}}$ ):

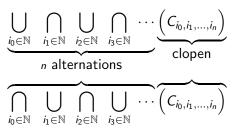
 $\underbrace{\bigcup_{i_0 \in \mathbb{N}} \bigcap_{i_1 \in \mathbb{N}} \bigcup_{i_2 \in \mathbb{N}} \bigcap_{i_3 \in \mathbb{N}} \cdots \underbrace{\left(C_{i_0, i_1, \dots, i_n}\right)}_{\text{clopen}}}_{n \text{ alternations}}$ 

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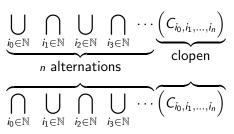


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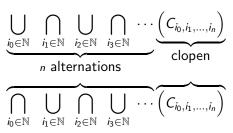


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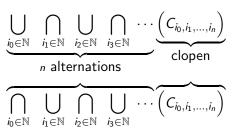


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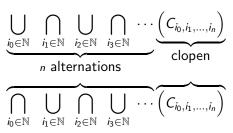
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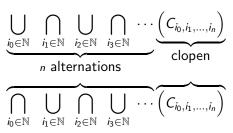
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 $\bigcup_{n\in\mathbb{N}} \mathbf{\Pi}_n$  is not a  $\sigma$ -algebra yet, so not all the Borel sets.

A  $\Sigma_{\alpha}$  subset of  $\mathcal{X}_{\tau}$  is a countable union of  $\Pi_{\beta}$  sets for  $\beta < \alpha$ 

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Definition:  $F: Copies(\mathcal{A})^2 \to \mathbb{N}^{\mathbb{N}}$  is an isomorphism function for  $\mathcal{A}$  if for any two copies,  $\mathcal{C}, \mathcal{D}$ , of  $\mathcal{A}$  with domain  $\mathbb{N}$ ,  $F(\mathcal{C}, \mathcal{D})$  is an isomorphism from  $\mathcal{C}$  to  $\mathcal{D}$ .

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Theorem: [M. 15] BorelRank(A) = BaireRank(A).

Recall: BorelRank(A) is the least  $\alpha$  such that Copies(A) is  $\Pi_{\alpha+1}$  in the Borel hierarchy.

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# Technique 3: Infinitary formulas for orbit and identification complexity.

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Theorem: [Scott 65] For every automorphism invariant set  $B \subset \mathcal{A}^k$ , there is an infinitary formula  $\varphi(\bar{x})$  such that  $B = \{\bar{b} \in \mathcal{A}^k : \mathcal{A} \models \varphi(\bar{b})\}.$ 

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# Infinitary 1st-order formulas

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We can measure identification complexity of  $\mathcal{A}$  in terms of the complexity of  $\psi_{\mathcal{A}}$ .

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 $\Pi^{in}_{\beta}$  for  $\beta < \alpha$ 

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Theorem: [Scott 65] For every automorphism invariant set  $B \subset \mathcal{A}^k$ , there is an infinitary formula  $\varphi(\bar{x})$  such that  $B = \{\bar{b} \in \mathcal{A}^k : \mathcal{A} \models \varphi(\bar{b})\}$ .

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[Lopez Escobar 65, Vaught 75]

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Consider the order  $(\mathbb{Q}; <)$ : •  $(a_1, ..., a_k) \cong (b_1, ..., b_k) \iff (\forall i, j < k) \ a_i < a_j \leftrightarrow b_i < b_j.$ 

Consider the order ( $\mathbb{Q}$ ; <):

- $(a_1,...,a_k) \cong (b_1,...,b_k) \iff (\forall i,j < k) \ a_i < a_j \leftrightarrow b_i < b_j.$
- The automorphism orbit of (1/2, 7, 4) is defined by a formula  $\varphi(x_1, x_2, x_3) \equiv x_1 < x_3 \land x_3 < x_2$ .

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#### Consider the order ( $\mathbb{Z}$ ; <):

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 $Rank(\mathbb{Z}; <) = 2.$ 

# $Rank(\cdot)$ is a robust measure of complexity

Theorem: [M. 15] Let  $\alpha$  be an ordinal and  $\mathcal{A}$  a countable structure. The following are equivalent:

- Orbit complexity
  - $\forall \bar{a} \in A^{<\mathbb{N}} \exists \bar{a}' \in A^{<\mathbb{N}} \forall \bar{b}, \bar{b}' \in A^{<\mathbb{N}}, \quad \bar{a}\bar{a}' \leq_{\alpha} \bar{b}\bar{b}' \Rightarrow \bar{a} \cong_{\mathcal{A}} \bar{b}.$
  - All automorphism orbits in  $\mathcal{A}$  are  $\Sigma^{in}_{\alpha}$ -definable.
- Identification complexity
  - The set of copies of  $\mathcal{A}$  is  $\Pi_{\alpha+1}$  in the Borel hierachy.
  - There is a  $\prod_{\alpha+1}^{in}$  sentence uniquely identifying  $\mathcal{A}$ .
- Isomorphism complexity
  - $\mathcal{A}$  has an isomorphism function that is of Baire class  $\alpha 1$ .

Let  $Rank(\mathcal{A})$  be the least such  $\alpha$ .

= bfRank(A) = OrbitRank(A) = BorelRank(A) = SSRank(A) = BaireRank(A).