Scott complexity of countable structures

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Can we measure this complexity? How?

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Three tools used to measure complexity:

- Back-and-forth relations for orbit complexity.
- Borel/Baire complexity for identification and isomorphism complexity.
- Infinitary formulas for orbit and identification complexity.

The main theorem

Theorem: [M. 15] Let α be a succesor ordinal and $\mathcal{A}$ a countable structure. The following are equivalent:

- All automorphism orbits in $\mathcal{A}$ are Σ^{in}_{α}-definable.
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Other definitions of rank had been proposed: [Scott 65][Sacks 07][Ash–Knight 00]...
However, this is the first equivalence theorem.

Technique 1

Technique 1: Back-and-forth relations

for orbit complexity.

Ash-Knight's version, in the spirit of Scott's original definition.

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Thus, $\bar{a} \equiv_0 \bar{b} \iff \bar{a} \cong \bar{b}$, and $akRank(\mathbb{Q}; <) = 0$.

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$\bar{a} \cong \bar{b} \iff \bar{a} \leq_1 \bar{b}\ \&\ \bar{b} \leq_1 \bar{a}$. Therefore $akRank(\mathbb{Z}; <) = 2$.

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- $\bar{a} \leq_2 \bar{b} \iff \forall \bar{d} \in A^{<\mathbb{N}}\ \exists \bar{c} \in A^{<\mathbb{N}} \quad \bar{a}\bar{c} \geq_1 \bar{b}\bar{d}$.

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Transfinite recursion: $(\forall H\colon \mathcal{X}^{ordinals} \to \mathcal{X})\ (\exists F\colon ordinals \to \mathcal{X})$

$$F(\alpha) = H(F \restriction \{\beta : \beta < \alpha\})$$

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(Do not read:) In reality we use the back-and-forth rank:
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Lemma: [M. 15] *akRank*$(\mathcal{A}) \leq$ *bfRank*$(\mathcal{A}) \leq$ *akRank*$(\mathcal{A}) + 1$.

Technique 2: Borel/Baire complexity
for identification and isomorphism complexity.

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Equip $\mathcal{X}_\tau$ with the topology from $\mathbb{N}^{\mathbb{N}}$ given by the power of the discrete topology.

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They are built from the clopen sets using countable unions and intersections.

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We can measure identification complexity of $\mathcal{A}$ in terms of the Borel complexity of *Copies*$(\mathcal{A})$.

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Recall: *BorelRank*$(\mathcal{A})$ is the least α such that *Copies*$(\mathcal{A})$ is $\mathbf{\Pi}_{\alpha+1}$ in the Borel hierarchy.

Technique 3

Technique 3: Infinitary formulas

for orbit and identification complexity.

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Examples

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$Rank(\mathbb{Z}; <) = 2$.

$Rank(\cdot)$ is a robust measure of complexity

Theorem: [M. 15] Let α be an ordinal and $\mathcal{A}$ a countable structure.
The following are equivalent:

- Orbit complexity
  - $\forall \bar{a} \in A^{<\mathbb{N}}\ \exists \bar{a}' \in A^{<\mathbb{N}}\ \forall \bar{b}, \bar{b}' \in A^{<\mathbb{N}}, \quad \bar{a}\bar{a}' \leq_\alpha \bar{b}\bar{b}' \Rightarrow \bar{a} \cong_{\mathcal{A}} \bar{b}$.
  - All automorphism orbits in $\mathcal{A}$ are Σ^{in}_α-definable.
- Identification complexity
  - The set of copies of $\mathcal{A}$ is $\mathbf{\Pi}_{\alpha+1}$ in the Borel hierachy.
  - There is a $\Pi^{in}_{\alpha+1}$ sentence uniquely identifying $\mathcal{A}$.
- Isomorphism complexity
  - $\mathcal{A}$ has an isomorphism function that is of Baire class $\alpha - 1$.

Let $Rank(\mathcal{A})$ be the least such α.
$= bfRank(\mathcal{A}) = OrbitRank(\mathcal{A}) = BorelRank(\mathcal{A}) = SSRank(\mathcal{A}) = BaireRank(\mathcal{A})$.