Logic and Mathematics. An interaction via Model Theory.

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- There are a number of striking applications of Model Theory in Algebraic and Complex Geometry, Number Theory, Analysis, ... (Hrushovski, Pila-Wilkie, Denef, Ax,...)
- Why is Model Theory so effective? What is its potential?
- Some parallels and lessons from Physics.

Why is Logic efficient in Maths?

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Heisenberg profoundly influenced Physics by proclaiming that only the notions which are observable can be accepted for foundations of physics.

Similarly, **definable notions and structures** are in the centre of Mathematics.

Categoricity

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M is well-defined?

$$\mathbf{M} \longleftrightarrow \mathrm{Th}(\mathbf{M}) \quad ?$$

Categoricity in cardinality gives us:

$$\mathbf{M} \longleftrightarrow (\mathrm{Th}(\mathbf{M}), \mathrm{card}\,\mathbf{M}).$$

Morley – Shelah Theory

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Aside: the criteria

$$\mathbf{M} \longleftrightarrow (\mathrm{Th}(\mathbf{M}), \mathrm{card}\,\mathbf{M})$$

can be applied to **pseudo-finite** structures **M** to speak about categoricity in an (unobservable) pseudo-finite cardinality.

Univocal theories and Physics

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Einstein, in particular, considered it to be an indispensable property of a scientific theory. (See D.Howard, I.Toader and other modern philosophers of physics on this.)

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This **extends** (!?) in some useful form to Abstract Elementary Classes.

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A Tentative Theorem. *Categorical theories are classifiable. Under mild extra assumptions any rich enough categorical theory is reducible to a fragment of complex geometry, possibly non-commutative.*

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Remark. Interesting categorical theories of complex non-commutative geometry involve specific (potentially very large) finite structures.

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An algebraic way to construct a pseudo-finite structure is by taking an **ultraproduct** of finite structures $\mathbf{N}_i$, $i \in I$ *along an ultrafilter* $\mathcal{D}$ *on* I

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Such an $^*\mathbf{N}$ is pseudo-finite by Łoś.

From finite to continuous. Continuous model theory.

Definition. An L-structure $\mathbf{M}$ is a limit of a pseudo-finite L-structure ${}^*\mathbf{N}$ if there is a surjective L-homomorphism

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We can also write equivalently

$$\lim_{\mathcal{D}} : \mathbf{N}_i \to \mathbf{M}.$$

Note. Homomorphisms preserve positively definable relations, "equations".

Approximation of fields

Any locally compact field (such as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$) can be *compactified* by adding ∞.
$\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, ...

Theorem.
1. The compactified field $\bar{\mathbb{C}}$ is approximable by finite fields.

2. The compactified field $\bar{\mathbb{R}}$ is **NOT approximable by finite fields or finite rings.** Neither are any other locally compact fields.

Can $\mathbb{R}$ be finitely approximated?

Instead of finite (residue) ring $(\mathbb{Z}/m\mathbb{Z}; +, \cdot, 0, 1)$ consider a fancier structure with two separate universes

$$(\mathbb{Z}/m\mathbb{Z}_{\mathrm{left}}; +, 0) \text{ and } (\mathbb{Z}/m\mathbb{Z}_{\mathrm{right}}; +, 0)$$

which weakens the multipliction to the bi-linear map from the left to the right:

$$\mathrm{e} : (x, y) \mapsto x \cdot y, \quad \mathbb{Z}/m\mathbb{Z}_{\mathrm{left}} \times \mathbb{Z}/m\mathbb{Z}_{\mathrm{left}} \longrightarrow \mathbb{Z}/m\mathbb{Z}_{\mathrm{right}}.$$

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(Thus, $x \cdot y \cdot z$ is not available.) The **weak ring structure** on $\mathbb{Z}/m\mathbb{Z}$.

Weak ring from $\mathbb{R}$ and $\mathbb{C}$

Let

$$\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}.$$

Theorem. The limit of weak ring structures $\mathbb{Z}/m\mathbb{Z}$ is the *structure* $\mathbb{R}_{\mathrm{weak}}$ *on two universes* $(\bar{\mathbb{R}}, +, 0)$ *and* $(\mathbb{S}, \cdot, 1)$ *with the "bi-linear" map between*:

$$\mathrm{e} : (x, y) \mapsto \mathrm{e}^{2\pi ixy}.$$

$$\begin{array}{ccc} (\mathbb{Z}/m\mathbb{Z}; +, 0)_{\mathrm{left}} & & (\mathbb{Z}/m\mathbb{Z}; +, 0)_{\mathrm{right}} \\ \downarrow & & \downarrow \\ (\bar{\mathbb{R}}, +, 0) & & (\mathbb{S}, \cdot, 1) \end{array}$$

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Moreover, this representation of the structure on $\mathbb{R}$ fits into the usual setting of non-commutative (quantum) complex geometry.

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Important: This theorem assumes that the “unobservable” $M := m$ in $\mathbb{Z}/m\mathbb{Z}$ is “very divisible”, i.e. divisible by all standard k.

Example of Calculation. Time evolution operator for the quantum harmonic oscillator.

$$K^t := e^{-i\frac{\mathrm{P}^2+\mathrm{Q}^2}{\hbar}t}, \quad t \in \mathbb{R}.$$

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And we get the matrix element

$$\langle x_1 | K^t x_2 \rangle = \sqrt{\frac{1}{2\pi i \hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2)\cos t - 2x_1 x_2}{2\hbar \sin t}.$$

Calculations via path integral

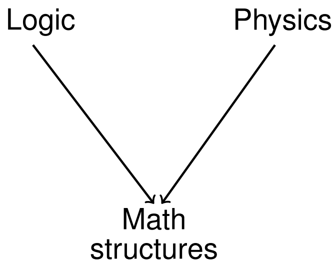
Need to evaluate $\lim\langle x_1 | I_\lambda x_2\rangle$, where

$$I_\lambda x = \sum_{x_1=-\frac{M^2}{2}}^{\frac{M^2}{2}} \dots \sum_{x_\lambda=-\frac{M^2}{2}}^{\frac{M^2}{2}} e^{i\sum a_{kj}x_k x_j} \Delta x_1 \dots \Delta x_\lambda, \quad K^t = \lim I_\lambda$$

$\Delta x = \frac{1}{M}$, $\Delta t = \frac{t}{\lambda}$.

The control over the pseudo-finite parameters allows to evaluate K^t correctly, **no need of renormalisation.**

Conclusions



Conclusions