Almost free groups and foundations of mathematics

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Outline

1. Main question
2. Elementary characterization of Whitehead groups
3. Finding a basis for a Whitehead group
4. Characterizing non-free Whitehead groups
5. Conclusions

1. Main question

Whitehead problem

- Whitehead(1951): Is every Whitehead group free?
- Stein(1951): Every countable Whitehead group is free.
- Shelah(1974): Whitehead problem is undecidable in ZFC.
  - Every constructible Whitehead group is free.
  - Assume MA $+ \neg$CH, there exists a non-free Whitehead group of cardinality $\aleph_1$.
- Question: What is the difference between Whitehead groups and free groups?

Foundations of mathematics

- Set theory as a foundation of mathematics:
  - Cantor, Dedekind, Zermelo, etc.
  - Set is the only basic object of mathematics.
  - Reduce the whole mathematics to the theory of sets.
- Arithmetic as a foundation of mathematics:
  - Kronecker, Poincare, Hilbert, etc.
  - Natural numbers and functions between natural numbers are the only basic object of mathematics.
  - Reduce almost the whole mathematics to the theory of natural numbers and sets of natural numbers, i.e. second-order arithmetic.

Set theory v.s. second-order arithmetic

- They coincide for ordinary mathematics such as calculus, abstract algebra, etc.
- They develop a theory of understanding infinite respectively:
  - set theory: various combinatorics and the hierarchy of higher infinite.
  - second-order arithmetic: various combinatorics and reverse mathematics: a scheme of slicing all the mathematical truth.
- Whitehead problem:
  - It is sensitive for models of set theory.
  - What about second-order arithmetic?

Main questions

- Question 1. What is the difference between Whitehead groups and free groups?
- Question 2. Is there a non-trivial theory about Whitehead groups in the framework of second-order arithmetic?

2. Elementary characterization of Whitehead groups

- In this talk, group will mean abelian group.
- Free group: a group with a basis.
- G is a Whitehead group if for every surjective homomorphism $\pi : P \to G$ with $\ker \pi$ isomorphic to $\mathbb{Z}$, there is a homomorphism $i : G \to P$ such that $\forall x \in G\ \pi(i(x)) = x$.
- Every free group is Whitehead.

- The language of homological algebra clarifies the structrual aspect of freeness and Whiteheadness;
- G is free if and only if $\forall A \ \mathrm{Ext}(G, A) = 0$;
- G is Whitehead if and only if $\mathrm{Ext}(G, \mathbb{Z}) = 0$.

- Cartan-Eilenberg Theorem: for every exact sequence

$$0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0,$$

there is exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}(C,G) \xrightarrow{\tilde{\tau}} \mathrm{Hom}(B,G) \xrightarrow{\tilde{\sigma}} \mathrm{Hom}(A,G) \to \\ \mathrm{Ext}(C,G) \to \mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G) \to 0; \\ 0 \to \mathrm{Hom}(G,A) \xrightarrow{\sigma'} \mathrm{Hom}(G,B) \xrightarrow{\tau'} \mathrm{Hom}(G,C) \to \\ \mathrm{Ext}(G,A) \to \mathrm{Ext}(G,B) \to \mathrm{Ext}(G,C) \to 0. \end{aligned}$$

The fundamental proposition for Whitehead groups

- Suppose G is a Whitehead group and H is a subgroup of G, then the followings are equivalent:
  - G/H is Whitehead;
  - every homomorphism from H to $\mathbb{Z}$ could be extended to a homomorphism from G to $\mathbb{Z}$.

Symbols and relations

- One could define a group from symbols and relations.
- Every group could be defined using symbols and relations.

Example: free groups G_f

Suppose $f : \omega \to \omega$,

- $$X = \{x_0, x_1, x_2, \dots ; y_0, y_1, y_2, \dots\}$$

  is the set of symbols;
- for each $i < \omega$,

  $$\sigma_i = x_{f(i)} - 2y_i$$

  is a relation on X.
- G_f is a free group, in fact,

  $$T = \{y_n : n \in \omega\} \cup \{x_n : \forall i \in \omega\ n \neq f(i)\}$$

  is a basis.

Example: a non-free group G_1

- $$Y = \{y_0, y_1, y_2, \dots\}$$

  is the set of symbols;
- for each $i < \omega$,

  $$\delta_i = 2y_{i+1} - y_i$$

  is a relation on Y.
- G_1 is not a free group.

- Suppose H is a group and there exists $a \in H$ such that

$$\forall n \in \omega \exists b \in H \ 2^n b = a,$$

then H is not free.

- The reason. Suppose T is a basis of H and

$$a = n_1 t_1 + \cdots + n_k t_k$$

where $t_1, \ldots, t_k \in T$. Suppose the index of 2 in the standard decomposition of $\gcd(n_1, \ldots, n_k)$ is l. Then there is no $b \in H$ such that $2^{l+1} b = a$.

Example: a free group G_2

- $$X = \{x_0, x_1, x_2, \ldots; y_0, y_1, y_2, \ldots\}$$

  is the set of symbols;
- for each $i < \omega$,

  $$\sigma_i = 2y_{i+1} - y_i - x_i$$

  is a relation on X;
- $\{y_0, y_1, \ldots\}$ is a basis of G_2.

Elementary characterization of Whitehead groups

- Suppose $\{\sigma_i : i \in I\}$ are independent relations on X, then the followings are equivalent:
  - the group defined from X and $\{\sigma_i : i \in I\}$ is Whitehead;
  - every linear system $\{\sigma_i = p_i : i \in I\}$ (where p_i's are abitrary integers) with unknowns in X has integeral solution.
- Whitehead group is torsion-free.
  - Subgroups of a Whitehead group are Whitehead.
  - $\mathbb{Z}/3\mathbb{Z}$ could be defined by a single symbol a and a relation $3a$. The equation $3a = 1$ has no integral solution, so $\mathbb{Z}/3\mathbb{Z}$ is not Whitehead.

The theory of Whitehead groups in RCA_0

- Definition of free groups and Whitehead groups.
- The fundamental proposition for Whitehead groups.
- Examples G_f, G_1 and G_2.
- The elementary characterization of Whitehead groups.
- Subgroups of a Whitehead group are Whitehead.
- Whitehead group is torsion-free.

3. Finding a basis for a Whitehead group

Pontryagin condition

- (The fundamental theorem on abelian groups)
  Every finitely-generated torsion-free group is free.
- Pontryagin condition: P and Q are finite subsets of G, we say Q covers P if every rational combination of elements of P is a linear combination of elements of Q. If every finite subgroup of G is covered by a finite subset, then we say G satisfies Pontryagin condition.
- For countable torsion-free group, freeness is equivalent to Pontryagin condition.
- Every Whitehead group satisfies Pontryagin condition.

- Stein's Theorem: Every countable Whitehead group is free.
- Pontryagin condition is preserved by any forcing extension.
- Any Whitehead group has a basis in the generic model in which its cardinality is collapsed.

Developing mathematics in models of second-order arithmetic

- Arithmetization of real numbers and analysis.
- Countable abstract structure such as groups, vector spaces etc.
- Turing ideal: $M \subseteq \mathcal{P}(\omega)$ and if $A_1, \ldots, A_k \in M$ and X is computable from $A_1, \ldots, A_k$, then $X \in M$.
- M is a Turing ideal if and only if $M \models \mathsf{RCA}_0$.
- Reverse Mathematics Program: Calibrating all mathematical theorems by which set existence axioms are using in their proofs.

$$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0.$$

- The phenomenon of lacking some kinds of functions in Turing ideals.
- For a given group, there may be no a function to decide whether a finite subset is independent.
- Freeness implies the existence of such a function.
- For a group satisfies Pontryagin condition, there may be no a function to assign a finite covering set to a finite set.
- Freeness implies the existence of such a function.

In RCA_0,

- For countable torsion-free group with a function to decide whether a finite subset is independent, freeness is equivalent to uniform Pontryagin condition: there is a function to assign a finite covering set to a finite set.
- Every Whitehead group satisfies Pontryagin condition.

- ACA_0 implies the existence of these functions.
- In ACA_0, Whitehead property is equivalent to freeness.
- Every arithmetical Whitehead group is free.
- If G is Whitehead in a Turing ideal M, then in an extended model M', G has a basis.

Two more deep results

- (Shelah 1974) Every constructible Whitehead group is free.
- (A joint work with Frank Stephan, Yang Yue and Yu Liang in 2018) Every computable Whitehead group is free.

Summary

- In some models of strong reduction or condensation property, every Whitehead group is free.
- For set theory universe or second-order arithmetic model, any Whitehead group has a basis in an extended model.

4. Characterizing non-free Whitehead groups

Investigating G_f in second-order arithmetic

(RCA_0) Suppose $f : \omega \to \omega$ is a one-to-one function. Then

- G_f is free if and only if the range of f exists;
- G_f is Whitehead if and only if the following proposition holds:

$$\Phi(f) : \forall g : \omega \to \{0,1\} \exists h : \omega \to \{0,1\} \forall i \in \omega \; h(f(i)) = g(i)$$

- Suppose G_f is Whitehead and $g : \omega \to \{0,1\}$.
- Define a linear system

$$\begin{cases} \check{x}_{f(0)} - 2\check{y}_0 = g(0) \\ \check{x}_{f(1)} - 2\check{y}_1 = g(1) \\ \qquad \dots \end{cases}$$

- $x_0, x_1, \dots, y_0, y_1, \dots$ is an integral solution.
- If $h(n)$ is the residue of x_n modulo 2, then

$$h(f(n)) = \text{the residue of } x_{f(n)} = g(n).$$

- WKL_0 implies $\forall f \Phi(f)$, thus G_f is Whitehead for all f.
- G_f is free for all f is equivalent to ACA_0.
- $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$ implies there is a non-free Whitehead G_f for some f.

Investigating G_E in set theory

Example: a non-free group G_3

- Fix a function $C : (\omega_1 \cap Lim) \times \omega \to \omega_1$ such that for each α, the sequence $C(\alpha, 0), C(\alpha, 1), \dots$ is strictly increasing and converges to α.
- $$X = \{x_\alpha : \alpha < \omega_1\} \cup \{y_{\delta,n} : \delta \in \omega_1 \cap Lim, n \in \omega\}$$

  is the set of symbols;
- for each $\delta \in \omega_1 \cap Lim$ and $i < \omega$,

  $$\sigma_{\delta,i} = 2y_{\delta,i+1} - y_{\delta,i} - x_{C(\delta,i)}$$

  is a relation on X.

G_3 is not free

- Subgroups $G^{(\gamma)}$ is generated by symbols from $\{x_\alpha : \alpha < \gamma\} \cup \{y_{\delta,n} : \delta \in \gamma \cap Lim, n \in \omega\}$;
- Suppose Y is a basis of G_3, then

$$\{\gamma < \omega_1 : Y \cap G^{(\gamma)} \text{ is a basis of } G^{(\gamma)}\}$$

  is a club.
- There is γ such that $G/G^{(\gamma)}$ is free, so $G^{(\gamma+1)}/G^{(\gamma)}$ is free.
- But $G^{(\gamma+1)}/G^{(\gamma)}$ is not free since $y_{\gamma,0}$ is dividable by any power of 2.

Example: G_E, subgroups of G_3

For $E \subseteq \omega_1$ such that there is no club contains in E,

- $$X = \{x_\alpha : \alpha < \omega_1\} \cup \{y_{\delta,n} : \delta \in E \cap Lim, n \in \omega\}$$

  is the set of symbols;
- for each $\delta \in E \cap Lim$ and $i < \omega$,

  $$\sigma_{\delta,i} = 2y_{\delta,i+1} - y_{\delta,i} - x_{C(\delta,i)}$$

  is a relation on X.

Then G_E is a subgroup of G_3.

Freeness of G_E

The followings are equivalent:

- G_E is free;
- E is nonstationary;
- for every $\delta \in E$, there is a $K_\delta \in \omega$ such that if $m > K_\delta, n > K_{\delta'}$, then $C(\delta, m) \neq C(\delta', n)$.

Whitehead property of G_E

The followings are equivalent:

- G_E is Whitehead;
- $\Psi(E)$: for every $c : E \times \omega \to \{0, 1\}$, there is $h : \omega_1 \to \{0, 1\}$ such that $h(C(\delta, i)) = c(\delta, i)$ holds for all $\delta \in E$ and sufficiently large i.

- Assume $\Psi(E)$ holds. To prove G_E is Whitehead, for any system

$$\begin{cases} \check{y}_{\delta,0} - 2\check{y}_{\delta,1} + \check{x}_{C(\delta,0)} = p(\delta,0), \\ \check{y}_{\delta,1} - 2\check{y}_{\delta,2} + \check{x}_{C(\delta,1)} = p(\delta,1), \\ \qquad \dots \\ \check{y}_{\delta,k} - 2\check{y}_{\delta,k+1} + \check{x}_{C(\delta,k)} = p(\delta,k), \\ \qquad \dots \end{cases}$$

- we could transfer it to

$$\begin{cases} (\check{y}_{\delta,0} - d(\delta,0)) - 2(\check{y}_{\delta,1} - d(\delta,1)) + \check{x}_{C(\delta,0)} = c(\delta,0), \\ (\check{y}_{\delta,1} - d(\delta,1)) - 2(\check{y}_{\delta,2} - d(\delta,2)) + \check{x}_{C(\delta,1)} = c(\delta,1), \\ \qquad \dots \\ (\check{y}_{\delta,k} - d(\delta,k)) - 2(\check{y}_{\delta,k+1} - d(\delta,k+1)) + \check{x}_{C(\delta,k)} = c(\delta,k), \\ \qquad \dots \end{cases}$$

such that every $c(\delta, i) \in \{0, 1\}$.

- There is $h : \omega_1 \to \{0, 1\}$ such that

$$h(C(\delta, i)) = c(\delta, i)$$

  holds for all $\delta \in E$ and $i > K_\delta$.
- Then
  - $x_\alpha = h(\alpha)$,
  - for $i > K_\delta$, $y_{\delta,i} = d(\delta, i)$,

  gives an integral solution.

- MA_{ω_1} implies $\forall E \Psi(E)$, thus G_E is Whitehead for all E.
- G_E is free for all E is equivalent to club filter over ω_1 is an ultrafilter, which contradicts Axiom of Choice.
- MA_{ω_1} implies there is a non-free Whitehead G_E for some E.

5. Conclusions

Free groups v.s. Whitehead groups

Free groups	Whitehead groups
a method to construct a splitting for every surjective homomorphism	the existence of a splitting for every surjective homomorphism
a method to find an integral solution for every linear system	the existence of an integral solution for every linear system
a method to find a covering set for every finite set	the existence of a covering set for every finite set
the range of f exists	$\Phi(f)$: for every $g : \omega \to \{0,1\}$, there is $h : \omega \to \{0,1\}$ such that$h(f(i)) = g(i)$ holds for all i.
E is nonstationary	$\Psi(E)$: for every $c : E \times \omega \to \{0,1\}$, there is $h : \omega_1 \to \{0,1\}$ such that $h(C(\delta,i)) = c(\delta,i)$ holds for all $\delta \in E$ and sufficiently large i.

hereditary set universe v.s. Turing ideal

hereditary set universe	Turing ideal
every countable Whitehead group is free	every arithmetical Whitehead group is free
every constructible Whitehead group is free	every computable Whitehead group is free
every Whitehead group is free in a generic extension	every Whitehead group is free in an extended model
MA + ¬CH implies there is a non-free Whitehead G_E for some E	$\mathrm{WKL}_0 + \neg\mathrm{ACA}_0$ implies there is a non-free Whitehead G_f for some f

Thank you.