The mechanisms and aims of classification in mathematics

Sean Walsh (UCLA)

Based on joint work with Tim Button (Philosophy, UCL)

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Two Main Questions

If one asks a mathematician what she is working on, often the answer will be that they are seeking to prove or refute a certain conjecture. Equally as often, though, the answer will be that they are seeking to classify a certain kind of mathematical object.

So proving and classifying seem like two central activities in mathematics. Philosophers of mathematics have written lots on proof, little on classification.

Here are some basic questions which one might ask about classification in mathematics. *What are the means and aims of classification? Can classification be reduced to proof?*

Two Main Questions, Continued

What are the means and aims of classification? This question can be made vivid by thinking about how we answer the corresponding question about proof. We are likely to say that *one* characteristic aim of proof is the extension of knowledge, and this aim is effected via formal deductions from known axioms. Even this brief answer suggests that understanding the activity of classification will involve two things: (i) identifying the aims of classification, and (ii) identifying the various mechanisms by which this aim is typically achieved. Given what we have said about proof, we might anticipate that answers to (ii) are accessible by inspection of mathematical texts, in a way in which answers to (i) need not be.

Can classification be reduced to proof? For example, one might wonder whether, for every classification program, there is a specific theorem such that the classificatory program is successful iff the theorem is successfully proven from accepted axioms. Maybe the reason that there's no separate literature on classification is that such a reduction is in fact possible.

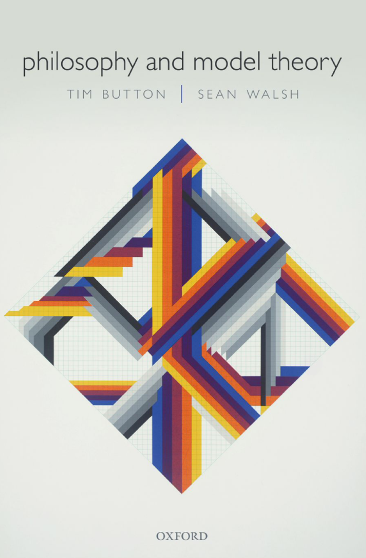
Shelah's Work on Classification

Shelah's work is obviously important for a lot of reasons: it resolved a well-known conjecture (Morley's conjecture), and the tools and concepts developed therein have been subsequently used in lots of applications of logic to other areas of mathematics.

But when you look at Shelah's own writings he seems to be suggesting a certain analysis of the notion of classification. We wanted to know whether that notion captured classification in mathematics more generally.

Outline

This material comes from sections 17.1-2 of:



Example from Algebra

In 1910, Steinitz classified the uncountable algebraically closed fields, such as the complex numbers. Steinitz's result says that two uncountable algebraically closed fields are isomorphic iff they have the same characteristic and the same cardinality.

This result follows from considerations regarding so-called transcendence bases which are included in most every introductory algebra textbook, e.g. [Lan02, §VIII.1], [Hun80, §VI.1], and esp. [Hun80, Theorem 1.12 p. 317].

Example from Topology

A second well-known example is the classification of compact connected surfaces. (For a discussion of the history, from Möbius in 1861 to Brahana in 1921, see e.g. [GX13, 151–7]).

This states: any compact connected surface is homeomorphic to the sphere, a connected sum of n-tori, or a connected sum of n-projective planes, and no two distinct surfaces on this list are homeomorphic to one another.

This result can also be stated as follows: two compact connected surfaces are homeomorphic iff they have the same Euler characteristic and either both are orientable or both are non-orientable.

Both of the invariants—Euler characteristic and orientability—can be represented as an integer. This result is mentioned in many introductory topology texts, e.g. [Kin93, 79, 107], [Law03, 120].

Example from Probability

A final celebrated example of classification is Ornstein's classification of isomorphism of Bernoulli shifts.

Suppose you have an n-sided die, with faces $1, \ldots, n$, and suppose you roll it once per minute, with no first roll, so that the sequences $\bar{x} = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots)$ correspond to individual histories of die-rolling.

Let Ω denote the set of all such sequences. Since we assumed that there is no first die-roll, there is a natural operation of 'fast-forwarding' on this space Ω given by moving the i^{th} entry in a sequence to the $(i+1)^{\text{th}}$ slot.

Where $p_1, \ldots, p_n$ are positive real numbers which sum to one, define a corresponding probability μ by saying that there is probability $p_{''j_1} \cdot p_{''j_2} \cdot \ldots \cdot p_{''j_k}$ of landing $''j_1$ on roll t_1, and landing $''j_2$ on roll t_2, ..., $''j_k$ on toss t_k, for distinct rolls.

Example from Probability, continued

The pair (Ω, μ) is called a *Bernoulli shift*, where the word ‘shift’ refers to the fast-forwarding operation. (In contexts where one is considering a wider class of operations, one might rather use ‘Bernoulli shift’ to refer to the triple formed by adding the fast-fowarding operation to the pair (Ω, μ).)

Where (Ω, μ) is a Bernoulli shift, its *entropy* is given by $-\sum_{i=1}^{n} p_i \log p_i$.

In 1970, Ornstein showed that any two Bernoulli shifts (Ω_1, μ_1) and (Ω_2, μ_2)—which may concern dice with different numbers of sides—are metrically isomorphic iff they have the same entropy ([Orn70], [Pet83, 281], [Rud90, §7]).

(To say that they are ‘metrically isomorphic’ is to say that there is a bijection $\Omega_1 \longrightarrow \Omega_2$ which preserves probabilities and respects fast-forwarding almost everywhere and whose domain and range need only be measure one sets ([Pet83, 4], [Rud90, 7])).

A General Mechanism

These three paradigmatic examples suggests the following general mechanism of classification. The initial data are given by a class C of mathematical objects and an equivalence relation E on C induced by a certain type of bijection between the objects.

The classification is then effected by identifying two further pieces of data: a class Inv of invariants, and an assignment of invariants in Inv to objects in C that respects equivalence. If one writes the assignment as $\iota : C \longrightarrow Inv$, then the requirement is that $E(X, Y)$ iff $\iota(X) = \iota(Y)$, as X, Y ranges over the classified objects in C.

We have seen this kind of general framework set out in [Ros11, 1252], [Gow08, 51].

C	E	Inv
uncountable alg. closed fields	isomorphism	characteristic and cardinality
compact connected surfaces	homeomorphism	Euler characteristic and orientability
Bernoulli shift	metric isomorphism	entropy

A General Mechanism: Variations

First, one might liberalise $E(X, Y)$ so that the equivalence relation need not be given by a bijection between X and Y. One does just this in the theory of Borel equivalence relations; see e.g. [Gao09]. A representative example is when X, Y are sequences of natural numbers and we define: $E_0(X, Y)$ iff there is some point after which X and Y agree.

Second, one might allow that $E(X, Y)$ is not an equivalence relation at all, but rather a metric-like similarity relation, which expressed that X, Y were close to one another in some sense. This is what happens in Gowers' [Gow00] notion of 'rough classification'.

So far as we can tell, everything we say in what follows is compatible with any of these modifications.

A Problem with the General Mechanism

The general mechanism described above is a good start for understanding classification programmes. However, it is excessively permissive.

To illustrate the point, let C be any class of objects C, with any equivalence relation E on them; put a well-order $\lhd$ on C by appealing to the Axiom of Choice and let $Inv \subseteq C$ consist of those elements of C which are the $\lhd$-least elements of their E-equivalence class; finally, let $\iota : C \longrightarrow Inv$ send each element to the unique element of C with which it is E-equivalent.

This will satisfy the minimum conditions stated above, but it does no useful classificatory work. Indeed, if such uninteresting appeals to the axiom of choice sufficed, then all classification problems would be immediately and trivially resolved.

A Problem with the General Mechanism, Continued

The problem arises here, because we have imposed no constraints on the nature of the invariants and their relations to the original class of objects.

Indeed, the issue here is similar to what happens by (mistakenly) regarding an *arbitrary* deduction from entirely *arbitrary* axioms as sufficient for engaging in serious mathematical proof.

Not only would this prevent you from accurately describing the activity of proof in mathematics; it would also blind you to the *aims* of proof.

An Associated Question, A Look at an Example

To deal with this, we must ask: *What distinguishes the invariants and assignments used in classification in mathematics from arbitrary invariants and assignments?*

To begin answering this, consider Ornstein's classification of isomorphism of Bernoulli shifts. The invariant here is entropy, which is given by $-\sum_{i=1}^{n} p_i \log p_i$. Evidently, this is an easily-calculated function of the tuple $(p_1, \ldots, p_n)$, and this tuple is itself prominent in the canonical presentation of the system (Ω, μ).

The Calculable Mechanism Thesis

This observation leads directly to the following thesis concerning how we should view mathematical classification. The invariants *Inv* and the function ι used in classifications in mathematics are such that:

1. ascertaining the particular invariant assigned to an object is easily calculable from a canonical presentation of that object (i.e. $\iota(X)$ is calculable from a canonical presentation of X); and,
2. the comparison of invariants can likewise be easily effected (i.e. it is easy to determine whether $\iota(X) = \iota(Y)$).

The Calculable Mechanism Thesis: Examples

This thesis resonates well with our two other paradigmatic examples.

In the example of compact connected surfaces, we think about the surface as 'triangulated', i.e. as broken up into a finite number of triangles, lines, and points, from which the Euler characteristic (for example) may be calculated.

In the example of algebraically closed fields, we conceive of the algebraically closed field as a set-sized structure which possesses a cardinality which may be easily ascertained. (This is in contrast to working with an all-encompassing 'universal domain', as is the default in some treatments of algebraically closed fields; see [Wei46, 242ff]).

The Calculable Mechanism Thesis: Evidence

But the strongest evidence for our thesis comes from the fact that mathematicians routinely talk about classification in patently computational terms, even in areas far removed from mathematical logic and the theory of computation. For instance, here is the beginning of a recent research monograph in differential topology:

> *A classification of manifolds up to diffeomorphism requires the construction of a complete set of algebraic invariants such that : [¶] (i) the invariants of a manifold are computable, [¶] (ii) two manifolds are diffeomorphic if and only if they have the same invariants, [. . .] ([Ran02, 1]).*

Similarly, in speaking of classifications, Gowers writes that 'as often as possible one should actually be able to establish when $\iota(X)$ is different from $\iota(Y)$. There is not much use in having a fine invariant if it is impossible to calculate' ([Gow08, 54]).

Potential Objection

It is worth noting that our thesis presupposes that canonical presentations are readily available to us (somehow). This is no surprise: the thesis would be fairly ineffectual otherwise, since proceeding by way of the canonical presentations might be just as difficult as enumerating all of the equivalence classes.

Now, someone might worry that a 'canonical presentation' can end up misidentifying the 'topic' of the relevant mathematical enquiry. For instance, in the topological case, one might have thought one was studying the *surfaces themselves*, and not their triangulations. Relatedly, one might worry that what counts as a 'canonical presentation' is historically contingent: a contemporary 'canonical presentation' of a surface might not have counted as 'canonical' in previous eras.

Response to Objection

Exactly the same issues pervade our ordinary ways of talking about proofs. In developing innovative proof techniques, one often appeals to new resources, and this can generate a concern that the topic has been changed.

For instance, Bolzano used the completeness of the real line to establish the intermediate value theorem, where previous mathematicians had sought to use considerations more closely related to the geometry of curves themselves ([Lüt03, 174–5]).

Likewise, students nowadays reason about products as sets of ordered pairs, or an object of a certain category, whereas previous eras might have rather talked about shapes of different dimensions.

Phenomena like these generate deeply interesting philosophical questions, such as *are proofs which do not introduce new concepts better?*, and *how should we think about theory change in mathematics?* (cf. [AD11], [Smi15]).

But, presumably everyone accepts that this phenomena is present in the activity of proving. It should not be surprising that the same is true of the activity of classifying.

The Aim of Classification

Now, the thesis is a proposal for how to think about the *activity* of classification within mathematics. But it also naturally suggests how to conceive of the *aim* of classification: classification is valuable because it leaves us better placed to calculate whether objects X and Y are (dis)similar, in the sense that we are better positioned to calculate whether $E(X, Y)$.

Of course, this does not tell us anything about *why* we might value the ability to distinguish similar from dissimilar mathematical objects. But this is just as it should be, for the answer to that general question will vary from case to case.

However, one background presupposition that is shared across all the examples is the following: contemporary mathematics is replete with a wide array of structures, and one task for mathematics is to provide a taxonomy of the most frequently encountered structures.

Completeness of Classifications

This view of classification also helps to explain some initially puzzling remarks about the kind of *completeness* which classifications sometimes give us. For instance, Steinitz motivates his classification of algebraically closed fields as follows:

> *Our program in this work is to obtain an overview of all possible fields and to ascertain their relations to one another with regard to their main features ([Ste10, 167]).*

We need to understand Steinitz's idea of *obtaining an overview* of all possible fields. At its most basic, we need to say why the truism 'all fields are isomorphic to the reals, or to some other field' fails to provide an overview in the relevant sense.

Our thesis suggests the following reading. The aim is that calculating invariants will provide an easy way to test for isomorphism of fields, where for each invariant there is also a simple example of a field with that invariant. More generally, classifications yield the relevant type of *completeness*, when it is possible both to describe all the invariants and provide an example of something with each invariant.

An Obstacle to Successful Classifications

Conversely, this suggests a way in which classification can be *unsuccessful*: namely, when it turns out that identifying and individuating the proposed invariants is just as hard as discerning the similarity of the classified objects in the first place.

In short: successful classification must employ invariants that are somehow 'simpler' than the objects to be classified. And it is notable that in our paradigmatic examples of classification, all the invariants were finite sequences of natural numbers, integers, or real numbers.

It would, of course, be lovely to have some greater understanding of what makes something fit to be an invariant; that is, to have a deeper understanding of the relevant notion of 'simplicity'. But, returning once again to the parallel with proof, this may well be just as hard to make headway on that as on the question of what makes something fit to be an axiom.

Relationship to Theorem-Proving

Recall the second basic question which we raised at the start of this section, namely: *Can classification be reduced to proof?*

After one has specified all the components of the mechanism—namely the equivalence relation, E, the invariants Inv, and the mapping ι—there is a clear theorem whose resolution is necessary for completing the classification.

However, providing that theorem is not, by itself, sufficient for success, since the mapping must be 'easily calculable', and one ought to be able to find explicit members of each equivalence class.

Relationship to Theorem-Proving, Continued

Moreover, the invariants are rarely given at the outset of the enquiry. A classification problem begins, instead, with the objects to-be-classified, C, and the similarity relation, E; and the task is to find the appropriate invariants. This is one good reason to resist offering a one-one ‘reduction’ of classification problems to specific theorems-to-be-proved.

There is also a second good reason. Whereas the proof of a theorem from accepted axioms is ultimately an all-or-nothing affair, the success of a classification program is a matter of degree. After all, one can debate the degree to which something is ‘easily calculable’, and one can debate the degree to which an element of each equivalence class has been explicitly described.

Shelah and the Morley Conjecture

Shelah's classification program culminated in the resolution of the Morley conjecture. Recall that Morley had conjectured that the number of non-isomorphic models of a complete theory, of a given uncountable cardinality, does not decrease as the cardinality increases (cf. [Fri75, 116])

Recall that $I(T, \kappa)$ is the number of isomorphism types of models of T, where we restrict our attention to models of size κ.

Morley had conjectured that if $\kappa \leq \lambda$ are both uncountable, then $I(T, \kappa) \leq I(T, \lambda)$. In the early 1980s, Shelah proved Morley's conjecture when T is countable; and the proof is contained in the second edition of his *Classification Theory and the Number of Nonisomorphic Models* ([She90]).

The Main Gap

The machinery from which Shelah derived the Morley conjecture also led to a result which Shelah called the *Main Gap Theorem*.

In a 1985 article, Shelah put the Main Gap Theorem this way ([She85, 228]; cf. [She90, 620], [HM85, 140], [Bal88, 3]):

Main Gap Theorem Let T be a complete theory in a countable language. Then one and only one of the following happens:

1. $I(T, \kappa) = 2^\kappa$, for all uncountable κ.
2. $I(T, \aleph_\gamma) \leq \beth_{\omega_1}(\max(|\gamma|, \omega))$ for all $\gamma > 0$; and in this case, T has a structure theory with countable depth.

Background to Main Gap

Even before his resolution of the Morley conjecture, Shelah had always indicated that the important idea in his work was a 'structure / non-structure dichotomy'. For instance, here is one statement of this idea:

> *At this stage, I will define stability theory as an attempt to give a classification of, and structure/non-structure theorems for elementary classes, and other related classes. An ideal structure theorem is a characterization (up to isomorphism) of each model in the class, by invariants, which are cardinals or sets of cardinals etc. [. . .] [¶] An ideal structure/non-structure theorem is the characterization of the classes which have a structure theorem, together with a proof of the complexity of the other classes ([She75, 241-242]; cf. also [She87, §1.1 p. 227], [She09b, §1.1 p. 154, immediately before Question 2.5]).*

This is from a 1975 paper that serves as an advance introduction to the first 1978 edition of Shelah's book. Interestingly, in the bibliography of the 1975 paper, Shelah proposed to call the 1978-book '*Stability* Theory and the Number of Nonisomorphic Models' (our emphasis).

Main Gap and Stability

Recall that T is λ-*stable* if the number of types in parameters of size $\leq \lambda$ is itself $\leq \lambda$; and that T is *stable* if it is λ-stable for some infinite λ.

The link between stability and classification emerged gradually Shelah's work. Indeed, the change in title of Shelah's, book from 'stability theory' to 'classification theory', was concomitant with the view that *classification* was central to the solution of the book's main problems. [She78, xii], [She90, xiv] writes: 'the change in the name of the book is not incidental, but a change in the point of view during the years in which it was written'.

All the theories which meet condition (2) in the Main Gap Theorem are stable and indeed are what is called *superstable*: they are λ-stable for all infinite cardinals $\lambda \geq$ the cardinality of the continuum.

Demarcating precisely which of the superstable theories meet condition (2) in the Main Gap Theorem is more difficult.

Cardinal-Like Invariants

We now defined by induction on α is the set of *cardinal-like invariants* $Inv_\alpha(\kappa)$ *of depth* α. Intuitively, α records the length of the iterative process, while κ records that these invariants are reserved for models whose underlying domain has cardinality κ. The recursive definition proceeds in three steps:

- $Inv_0(\kappa)$ is the set of all cardinals $\lambda \leq \kappa$, which we write just as the set $\{\lambda : \lambda \leq \kappa\}$
- $Inv_{\alpha+1}(\kappa)$ is the set of sequences of length less than or equal to the cardinality of the continuum, with each element of the sequence being a function $f : Inv_\alpha(\kappa) \longrightarrow \{\lambda : \lambda \leq \kappa\}$
- $Inv_\alpha(\kappa)$ is $\bigcup_{\beta<\alpha} Inv_\beta(\lambda)$, when α is a limit.

Finally, we define Inv_α to be the union of $Inv_\alpha(\lambda)$ as λ ranges over all infinite cardinals.

Shelah's Definition of Classifiable

This notion of cardinal-like invariance is the key component to Shelah's explication of 'having a structure theory' or 'being classifiable' ([She85, 228]; cf. [She87, §1.4 p.155, setting $\chi = 2^{\omega}$], [She09b, §2.9 p.25] [Bal87, 5]).

In particular, Shelah says that T *has a structure theory of depth* α if there is a function ι from the set of models of T to Inv_{α} such that:

1. if $\mathcal{M}$ has size κ then $\iota(\mathcal{M})$ is in $Inv_{\alpha}(\kappa)$, and
2. if $\mathcal{M}, \mathcal{N}$ are two models of T, then $\mathcal{M}$ is isomorphic to $\mathcal{N}$ iff $\iota(\mathcal{M}) = \iota(\mathcal{N})$.

Finally, we say that T *has a structure theory* if there is an α such that T has a structure theory of depth α. This is the definition of 'having a structure theory' which occurs in condition (2) of the Main Gap Theorem.

The Shelah Analysis of Classifiability

One way to then think about the Main Gap theorem is as offering us an analysis of classification, to wit:

1. theories are classifiable iff they do not have 'too many models'
2. when theories are classifiable, we can characterise each of their models up to isomorphism by invariants which are 'cardinal-like'.

There's a couple different ways to think about why it's appropriate to attribute this to Shelah.

First, the 'non-structure' side of his original 'dividing lines' are always defined by 'having too many models.' That is, it seems he's stipulatively identifying 'non-structure' with 'having too many models'.

Second, kind of looks like he takes 'cardinal-like invariant' to be a maximally general notion of invariant. One can then show that things on the 'wild' side of the Main Gap can't be classified with cardinal-like invariants:

Proposition Given GCH, if there is infinite κ such that $I(T, \lambda) = 2^\lambda$ for all $\lambda \geq \kappa$, then T does not have a structure theory.

See: [She85, 228]; cf. [She87, §1.6 p.155][25–6 immediately below Corollary 2.12]Shelah2009ab.

Two Different Conceptions of Classifiability

Calculable mechanism: to classify is to provide ‘easily calculable’ invariants, for classes of objects under various equivalence relations, and explicit examples of each member of equivalence class. Then classes of objects are classif*iable* if such a classification can be provided.

Few (equivalence) classes: theories are classifiable iff they do not have ‘too many models’. Derivatively, classes of models of a given theory are classifiable if they are not too many in number (up to isomorphism).

Caveat 1: Obviously the intended application of *few classes* is to the models of a complete first-order theory in a countable language, whereas *calculable mechanisms* is intended to cover notions of classification from algebra, topology, etc. But we can mitigate this difference by either restricting *calculable mechanisms* to complete first-order theories, or by taking *few classes* to be proposing something about classification in a more general sense. Of course, Shelah and others have worked on extending his program out of the non first-order case.

Two Different Conceptions of Classifiability, continued

Calculable mechanism: to classify is to provide 'easily calculable' invariants, for classes of objects under various equivalence relations, and explicit examples of each member of equivalence class. Then classes of objects are classif*iable* if such a classification can be provided.

Few (equivalence) classes: theories are classifiable iff they do not have 'too many models'. Derivatively, classes of models of a given theory are classifiable if they are not too many in number (up to isomorphism).

Caveat 2: We're going to start describing arguments for and against *few classes*. We're doing that because we take it as kind of obvious that there's an *intensional* difference between the two proposals: just because invariants are easy to calculate doesn't mean that there are few invariants, and vice-versa. But this might leave open that in many mathematical settings there's a demonstrable *extensional* equivalence between the two proposals. We'll mention a specific setting later in the section.

Hodges on *Few classes*

Hodges notes that group theorists have provided an apparently successful classification of 'totally projective abelian p-groups', *despite* the fact that there are 2^λ of them in any uncountable cardinality λ ([Hod87, 231, 221]).

By contrast, Problem 51 in the 1973 version of Fuchs' *Infinite Abelian Groups* was to 'Characterize the separable p-groups by invariants' ([Fuc73, 55]). In 1974, Shelah showed that for regular uncountable λ there are 2^λ non-isomorphic separable p-groups of cardinality λ ([She74, Theorem 1.2 pp.245–6]).

Shelah wrote of this that 'the proof indicates to me that separable p-groups cannot be characterized by any reasonable set of invariants. (This answers Problem 51 of Fuchs [. . .])' ([She74, 244]). In the later editions of Fuchs' book, Problem 51 no longer appears; in its place, special cases of Shelah's result are given ([Fuc15, 332–3])

Hrushovski on *Few classes*

The argument begins from the reasonable supposition that there is no sense in which *all models* can be classified (cf. [Hod87, 232]).

Now, if we are inclined to restrict attention to countable signatures anyway, then we may as well view all structures under consideration as structures in a maximally generous signature with countably many constant symbols and countably many relation and function symbols of all numbers of places.

And for a given infinite cardinality κ, there are exactly 2^{κ}-many non-isomorphic models in that signature. By appeal to the premise that there is no reasonable sense in which one can usefully classify all models of a given infinite cardinality, one then concludes that the same fate befalls any theory which has just as many models.

More on Hrushovski for *Few classes*

One premise of the Hrushovski argument seems to be: if you can't classify the ABC's, and there's exactly as many of those (up to equivalence) as there are XYZ's, then you can't classify the XYZ's.

Not obvious this is true. Potential counterexamples:

- If the continuum hypothesis holds, then there are exactly 2^{ω} countable well-orders (up to isomorphism). But in very elementary set theory, one learns various methods for determining whether two well-orders are isomorphic, or if rather one is isomorphic to a proper initial segment of the other.
- Consider Cauchy sequences of rationals, whose equivalence classes are of course just reals. These are classifiable (it seems), despite there being lots of them.

The Other Direction

In the last slides, we've been looking at arguments for and against one direction of *Few classes*, namely: if the objects are classifiable, then there are few equivalence classes.

The other direction of *Few classes*, is: if there are few equivalence classes, then the objects in question are classifiable.

The *calculable mechanism* view predicts this is false in the abstract: imagine a case where there were only countably many equivalence classes but determining which case you are in is really rather difficult.

But it's hard to come up with mathematically natural example. For a less-than-natural example: try classifying recursively enumerable subsets of the natural numbers up to Turing equivalence (this classification problem computes the fourth Turing jump).

Rapprochement (?)

In generalised descriptive set theory, the idea is to view models of uncountable cardinality κ as points in a topological space. Each model in a countable signature whose underlying domain has cardinality κ can be naturally coded as a function from κ to κ, and the underlying domain of the topological space is the set of all such functions.

There is a measure of complexity on subsets of this space where: open sets are least complex; the Borel sets (those obtained from the opens through complementation and κ-sized unions) are more complex; and the analytic sets (those formed from projection over closed sets) are yet more complex. This measure of complexity can be extended naturally to the product spaces, so that it makes sense to ask after the complexity of relations between structures.

Rapprochement (?), continued

Väänänen is one of the first to study model theory from this perspective, and he writes: ‘It turns out that stability theory and the topological approach proposed here give similar suggestions as to what is complicated and what is not’ ([Vää08, 117]).

This has recently been confirmed in a startling way by results of Friedman–Hyttinen–Kulikov. They show that for certain infinite cardinals κ, the relation of isomorphism between models of cardinality κ is Borel iff it falls on the ‘has a structure theory’ side of Shelah's Main Gap Theorem ([FHK14, Theorem 63 p.55]).

This shows that when we take a suitably expansive sense of ‘calculable’, Shelah's notion of classification aligns extensionally with ours.

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