Updates on Higher Degree Theory

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Contents

Turing degrees

- Recursion theory, now called **computability theory**, is a branch of mathematical logic that originated in the 1930s with the aim of studying recursive/computable functions and Turing degrees.
- Recursion theory has traditionally focused on relative computability.
- For $A, B \subseteq \omega$, $A \leqslant_T B$ if both A and $\omega \backslash A$ can be recursively computed from B.

  $\leqslant_T$ induces an equivalence relation – Turing degrees, $\boldsymbol{a} = [a]_T$

  Turing degrees $= \Delta^0_1$-degrees.
- Turing degrees, together with Turing reducibility, $(\mathscr{D}, \leqslant_T)$, form an upper semi-lattice, i.e. a partially ordered set that has a least upper bound for any nonempty finite subset.

Structure of the Turing degrees

Here are some properties of $(\mathscr{D}, \leqslant_T)$.

- $\forall \boldsymbol{a} \neq \mathbf{0}$, $\exists \boldsymbol{b}\,(\boldsymbol{a}, \boldsymbol{b}$ are incomparable$)$.
  $\implies \leqslant_T$ is not a linear order.
- There are $2^{\aleph_0}$ many pairwise incomparable Turing degrees.
  $\implies$ the width of $(\mathscr{D}, \leqslant_T) = 2^{\aleph 0}$.
- There are minimal degrees.[1]
  $\implies \leqslant_T$ is not dense.
- Every two degrees $\boldsymbol{a}, \boldsymbol{b}$ have the least upper bound, $\boldsymbol{a} \oplus \boldsymbol{b}$. But there are pairs of degrees with no greatest lower bound. Thus $(\mathscr{D}, \leqslant_T)$ is only an upper semi-lattice.
- Every countable partially ordered set can be embedded in the Turing degrees.
- No infinite, strictly increasing sequence of degrees has a least upper bound. So $(\mathscr{D}, \leqslant_T)$ has height $\aleph_1$.

[1] A degree $\boldsymbol{a}$ is minimal (w.r.t. $\mathbf{0}$) if $\boldsymbol{a} > \mathbf{0}$ and $\neg\exists \boldsymbol{b}\,(\mathbf{0} <_T \boldsymbol{b} <_T \boldsymbol{a})$.

- For any set X, the notation X' denotes the set of indices of oracle machines that halt when using X as an oracle.
- The set X' is called the Turing jump of X. The Turing jump of a degree $\boldsymbol{X}$ is defined to be the degree $\boldsymbol{X}'$.
- A key example is $\mathbf{0}'$, the degree of the halting problem.

Properties involving the jump:

- For any degree $\boldsymbol{a}$, there is a degree $\boldsymbol{b}$ s.t. $\boldsymbol{a} < \boldsymbol{b}$ and $\boldsymbol{b}' = \boldsymbol{a}'$.
- There is a sequence $\langle \boldsymbol{a}_i : i < \omega \rangle$ of degrees s.t. $\boldsymbol{a}'_{i+1} \leqslant \boldsymbol{a}_i$, $\forall i$.
- (Posner-Robinson, 1981). For any degree $\boldsymbol{a}$, $\boldsymbol{a} \neq \mathbf{0}$ iff there is a degree $\mathbf{g}$ s.t. $\boldsymbol{a} \oplus \boldsymbol{g} = \boldsymbol{g}'$.
- (Shore-Slaman, 1999). The jump operator is 1^{st}-order definable in the structure $(\mathscr{D}, \leqslant_T)$.
- (Simpson, 1977). The 1^{st}-order theory of $(\mathscr{D}, \leqslant_T)$ is "equivalent" to the theory of 2^{nd}-order arithmetic.

Generalizations of Turing degrees

- Turing reduction is the simplest form of definability reduction. The study of degree notions is naturally extended to higher degrees of definability reduction, such as arithmetic degrees, hyperarithmetic degrees, higher degrees of projective hierarchies, even constructible degrees, degrees induced by inner model operators, etc. Many of these are studied in Descriptive Set Theory.
- One can also extend the notion of Turing degree on subsets of ω to subsets of large ordinals. This program was initiated by Sacks, and is called α-recursion theory. However, this was mainly studies within the constructible universe L, the smallest inner model of set theory.

Contents

Higher Degree Theory

- Classical recursion theory studies the structure of Turing degrees.
- It has been extended/generalized to
  - higher levels of computability (e.g. hyperarithmetic degrees)
  - higher ordinals/cardinals (e.g. α-recursion, i.e. generalized Turing degrees at α)
  - even both (e.g. generalized hyperarithmetic α-degrees)

  However these do not go beyond ZFC.
- Recent developments reveal some exciting connections between large cardinals and degree structures at uncountable cardinals, in particular, strong limit singular cardinals of countable cofinality.
- The program of higher degree theory
  - studies definability degree structures
  - focus on the connection between large cardinals and degree structures.

Gödel's constructible universe L

For each set X, let $\mathscr{P}_{\text{Def}}(X)$ denote the set of all $Y \subseteq X$ such that Y is definable in the structure $(X, \in)$ (from parameters in X).

Gödel's constructible universe L

Define L_α by induction on α as follows:

- $L_0 = \varnothing$
- $L_{\alpha+1} = \mathscr{P}_{\text{Def}}(L_\alpha)$.
- If λ is a limit ordinal, then $L_\lambda = \bigcup_{\beta<\lambda} L_\beta$.

L is the class of all sets X such that $X \in L_\alpha$ for some α, i.e.

$$L = \bigcup_{\alpha \in \text{Ord}} L_\alpha.$$

L-like models

$$L(A):\quad \begin{aligned} L_0(A) &= \mathrm{TC}(\{A\}) \\ L_{\alpha+1}(A) &= \mathscr{P}_{\mathrm{Def}}(L_\alpha(A)) \\ L_\lambda(A) &= \bigcup\{L_\alpha(A) \mid \alpha < \lambda\} \qquad \text{(for limit } \lambda > 0) \end{aligned}$$

$$L[A]:\quad \begin{aligned} L_0[A] &= \varnothing \\ L_{\alpha+1}[A] &= \mathscr{P}^A_{\mathrm{Def}}(L_\alpha[A]) \\ L_\lambda[A] &= \bigcup\{L_\alpha[A] \mid \alpha < \lambda\} \qquad \text{(for limit } \lambda > 0) \end{aligned}$$

where $\mathrm{TC}(x)$ is the smallest transitive set containing x,
$\mathscr{P}^A_{\mathrm{Def}}(X) = \{Y \subseteq X \mid Y \text{ is definable over } (X, \in, A \cap X)\}$

Generalize Turing degree

- A is Turing reducible to B, i.e. $A \leqslant_T B$ iff $A \in \Delta^0_1(B)$, where $\Delta^0_1(B)$ is the collection of subsets of ω that is Δ^0_1-definable over the structure $(\omega, \in, B)$.
- A is arithmetically reducible to B if $A \in \Delta^0_\omega(B)$, where $\Delta^0_\omega(B) = \bigcup_{n<\omega} \Delta^0_n(B)$. This is equivalent to say that $A \in \mathscr{P}^B_{\mathrm{Def}}(\omega) = L_{\omega+1}[B] \cap \mathscr{P}(\omega)$.
- A is hyperarithmetically reducible to B if $A \in \Delta^1_1(B)$, where $\Delta^1_1(B)$ is the collection of subsets of ω that is closed under recursive operators. $\Delta^1_1(B) = L_{\omega_1^B}[B] \cap \mathscr{P}(\omega)$.

In general, one can define

Let Γ be a reasonable theory in the language of set theory, and $A, B \subset \omega$. A is Γ-reducible to B if $A \in M[B] \cap \mathscr{P}(\omega)$, where $M[B]$ is the smallest model of Γ that contains $\{B\}$.

Zermelo degrees

Let $\mathsf{Z} = \mathsf{ZF} - \text{Replacement}$. Use Z-degrees to illustrate the idea.

Definition

Suppose λ is a limit of at least strongly inaccessible cardinals. Fix a well-ordering $w : H(\lambda) \to \lambda$. For $a, b \subset \lambda$:

- $M[a]$ denotes the minimal Z-model of the form $L_\alpha[w][a]$, $\alpha > \lambda$. Let α_a, Z-ordinal for a, denote the height of $M[a]$.
- $a \leqslant_{\mathsf{Z}} b$ if $M[a] \subseteq M[b]$. $a \equiv_{\mathsf{Z}} b$ if $a \leqslant_{\mathsf{Z}} b$ and $b \leqslant_{\mathsf{Z}} a$
- Write $\boldsymbol{a}$ for the degree of a, the $\equiv_{\mathsf{Z}}$-equivalence class of a.
- $J_{\mathsf{Z}}(a)$, Z-jump of G, is the theory of $M[a]$. It can be coded by a subset of λ.

A list of questions

A list of degree theoretic questions.

1. (Post Problem). Are there incomparable degrees, i.e.
$$\neg(\boldsymbol{a} \leqslant \boldsymbol{b}) \wedge \neg(\boldsymbol{b} \leqslant \boldsymbol{a})?$$
2. (Minimal Cover). Given $\boldsymbol{a}$, is there a $\boldsymbol{b}$ minimal w.r.t. $\boldsymbol{a}$, i.e.
$$\boldsymbol{a} < \boldsymbol{b} \wedge \neg\exists \boldsymbol{c}(\boldsymbol{a} < \boldsymbol{c} < \boldsymbol{b})?$$
3. (Posner-Robinson). Is it true for co-λ many $x \subset \lambda$ that
$$(\exists G)[\boldsymbol{x} \oplus \boldsymbol{G} \equiv_{\mathsf{Z}} \boldsymbol{J}_{\mathsf{Z}}(\boldsymbol{G})]?$$
4. (Degree Determinacy). Is $\mathrm{Det}_\lambda(\text{Z-Deg})$ true?[2]

For $(\mathscr{D}, \leqslant_T)$, the answers are all Yes.

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[2] $\mathrm{Det}_\lambda(\text{Z-Deg})$: Every Z-degree invariant subset of $\mathscr{P}(\lambda)$ either contains a cone or is disjoint from a cone. Here

- A set $A \subset \mathscr{P}(\lambda)$ is Z-degree invariant if $a \in A \Rightarrow \boldsymbol{a} \subset A$.
- A cone is a set of the form $C_a = \{b \mid a \leqslant b\}$.

Degree structures in forcing extensions

Not very much of degree structures at uncountable cardinals can be determined by ZFC alone, even with large cardinals.

Example

- Assume $\mathrm{ZFC} + \mathrm{GCH}$ and plus some large cardinal assumption, say a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$ plus a measurable cardinal $\kappa' > \kappa$.
- With a small forcing, one can arrange that in $V[G]$: $\kappa = \aleph_\omega$, GCH remains true below $\aleph_\omega$, $2^{\aleph_\omega} = \aleph_{\omega+2}$ while the measurability of κ' is preserved (This combines results of Woodin and Gitik).
- But in $V[G]$: the Zermelo degree posets at $\aleph_\omega$ can not be well ordered, as every degree has only $\aleph_\omega$ many predecessors in the degree partial ordering.

Studying higher degrees in canonical models

On the other hand, fine structure models provide very complete settings for answering most questions. In these models, objects are constructed in a well organized manner – the rigidity of these models clears out many structural chaos, and makes the impacts of large cardinal axioms to the structure of degrees more evident.

Consider degrees at λ inside an L-like models.

- $\mathrm{cf}(\lambda) = \lambda$, i.e. λ is regular. *Not very interesting*.
  Most degree theoretic constructions at ω can be generalized to such λ.

[The two main techniques used in recursion theory are priority and forcing arguments. Most of the combinatorics of ω needed for these techniques to run at regular λ can be obtained by assuming $\lambda^{<\lambda} = \lambda$, which follows from GCH.]

At singular cardinals with uncountable cofinality

- $\mathrm{cf}(\lambda) > \omega$, e.g. $\lambda = \aleph_{\omega_1}$. *Nothing interesting left.*

Theorem (Sy Friedman, 81) ($V = L$)

The $\aleph_{\omega_1}$-degrees are well-ordered above some degree.

Sy Friedman's trick is the analysis of *stationary subsets* of $\mathrm{cf}(\lambda)$. His argument works in all $L[E]$-like inner models for most definablity degree notions (that are coarser than α degree) at any singular cardinals of uncountable cofinality.

Theorem (Sy Friedman) ($V =$ any fine structure model)

The generalized degrees at singular cardinals of uncountable cofinality are well-ordered above some degree.

Pictures in L

- $\mathrm{cf}(\lambda) = \omega$, e.g. $\lambda = \aleph_\omega$. *Where the fun is.*

Theorem (S., 2015) ($V = L$)

If $\mathrm{cf}(\lambda) = \omega$, then Z-degrees at λ are well-ordered above some degree. In particular, Z-degrees at $\aleph_\omega$ is well-ordered.

ANSWERS TO THE LIST. (above special degrees)

Post Problem	No.
Minimal Cover	Yes. "No" for > 1 minimal covers.
Posner-Robinson	No.
Degree Determinacy	No.

Covering for L

A key ingredient of the argument is

Covering Lemma for L. (Jensen, 74)

Assume $\neg\exists 0^\sharp$. Then every set $x \subset \mathrm{Ord}$ is covered by a $y \in L$, with $|y| = |x| + \omega_1$.

Remark

For inner models above L and below $L[\mu]$, the minimal inner model for one measurable cardinal, the same argument applies, since their Covering Lemmas are of the same form.

There is a little wrinkle in canonical models for finitely many measurable cardinals – the covering lemma for $L[\mu]$ have different format.

Degrees in $L[\mu_0, \ldots, \mu_n]$

Covering Lemma for $L[\mu]$. (Dodd-Jensen, 82)

Assume $\neg\exists 0^\dagger$, but there is an inner model $L[\mu]$. Let $\kappa = \text{crit}(\mu)$. Then for every set $x \subset \text{Ord}$, one of the following holds:

1. Every set $x \subset \text{Ord}$ is covered by a $y \in L[\mu]$, with $|y| = |x| + \omega_1$.
2. $\exists C$, Prikry generic over $L[\mu]$, s.t. every set $x \subset \text{Ord}$ is covered by a $y \in L[\mu][C]$, with $|y| = |x| + \omega_1$.
   Such C is unique up to finite difference.

The argument for L can be adapted to yield the same picture at every singular cardinal of countable cofinality – Z-degrees are well-ordered above some point.

Degrees in $L[\bar{\mu}]$

Theorem (S., 2015)

Assume $V = L[\bar{\mu}]$, where $\bar{\mu} = \langle \mu_n : n < \omega \rangle$, each μ_n is a measure on κ_n. Let $\kappa_\omega = \sup_n \kappa_n$. Suppose λ is a singular cardinal of countable cofinality.

1. *If $\lambda \neq \kappa_\omega$, then the Z-degrees at λ are wellordered above the degree of $\langle \kappa_n : n < \omega \rangle$ (in fact any singularizing degree);*
2. *If $\lambda = \kappa_\omega$, consider only Z-degrees at λ above the degree of $\bar{\mu}$.*
    - *Z-degrees at λ above the degree of $\bar{\mu}$ are prewellordered via their Z-ordinals.*
    - *Let A_η be a subset of λ that codes the first η many Z-ordinals, and $\mathcal{C}_\eta$ be the set of $L_{\alpha_\eta}[\bar{\mu}]$-generic Prikry sequences for $\mathbb{P}_{\bar{\mu}}$. Then Z-degrees at λ (above the degree of $\bar{\mu}$) whose Z-ordinals equal to the η-th Z-ordinal are exactly the degrees given by*

$$A_\eta \oplus \mathcal{C}_\eta = \{(A_\eta, C) \mid C \in \mathcal{C}_\eta \cup \{\varnothing\}\}.$$

Pictures in $L[\bar{\mu}]$

In addition, it's not difficult to see the following

- there are infinite descending chains of degrees.
- there are 2^ω many pairwise incomparable degrees.

Recently, Zhang Teng obtained a few more properties of this degree structure.

Theorem (Zhang)

Assume $V = L[\bar{\mu}]$ and $\lambda = \sup_n \kappa_n$ et al as in the previous theorem.

1. *There is an anti-chain of size 2^λ in the Z-degrees at λ.*
2. *There is an independent set of size 2^λ in the Z-degree at λ.*

Pictures in $L[\bar{\mu}]$

ANSWERS TO THE LIST. (at $\lambda = \sup_n \kappa_n$)

Post Problem	Yes. $\exists$ antichain of size 2^λ
Minimal Cover	No.
Posner-Robinson	No.
Degree Determinacy	No.

Picture in $L[\mathcal{U}]$ for $o(\kappa) = \kappa$

Theorem (Yang, 2011)

In the canonical model for a sequence $\langle \kappa_n : n < \omega \rangle$ such that $o(\kappa_n) = \kappa_{n-1}$, ($\kappa_{-1} = 1$), at $\lambda = \sup_n \kappa_n$ there is a Z-degree $\boldsymbol{D}$ such that there are minimal Z-degrees w.r.t. $\boldsymbol{D}$.

Yang Sen's argument can be modified to show that

Theorem

Let the model and λ are as above. There are 2^λ minimal degrees w.r.t. $\boldsymbol{D}$, hence 2^λ many pairwise incomparable degrees.

Picture in $L[\mathcal{U}]$ for $o(\kappa) = \kappa$

In this model the answers are:

Post Problem	Yes. $\exists\, 2^\lambda$ many pairwise incomp. degrees.
Minimal Cover	Yes. $\exists\, 2^\lambda$ many minimal covers (over $\boldsymbol{D}$).
Posner-Robinson	very plausible to be “No”.
Degree Determinacy	very plausible to be “No”.

Picture from the top, I_0

Definition

$I_0(\lambda)$ is the following assertion: There exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $\mathrm{crit}(j) < \lambda$.

Theorem (S, 2015)

Assume ZFC $+ I_0(\lambda)$. *Then*

1. *For almost all (co-λ many) $X \subset \lambda$,*

$$(\exists G \subset \lambda\ )\,[\boldsymbol{x} \oplus \boldsymbol{G} \equiv_{\mathsf{Z}} \boldsymbol{J}_{\mathsf{Z}}(\boldsymbol{G})].$$

2. *With a mild techincal assumption,*[a] *we have*

$$L(V_{\lambda+1}) \models \neg\mathrm{Det}_\lambda(\text{Z-Deg}).$$

[a] Suppose in V_λ, κ_0 is supercompact, and it supercompactness is indestructible by κ_0-directed posets.

Pictures in I_0 models

The answers to the list are as follows:

Post Problem	Yes. $\exists\, 2^\lambda$ many pairwise incomp. degrees.
Minimal Cover	Yes. $\exists\, 2^\lambda$ many minimal covers.
Posner-Robinson	Yes.
Degree Determinacy	Consistently "No".

Complexity of degree structures

Theorem ($V = L[\bar{\mu}]$)

Suppose $\lambda > \omega$ and $\mathrm{cf}(\lambda) = \omega$.

1. *If λ is not a limit of measurable cardinals, then the Z-degrees at λ is well ordered above some singularizing degree;*
2. *if λ is a limit of measurable cardinals, then there are incomparable Z-degrees at λ.*

Remark.

The complexity of degree structures at certain cardinals reflects the strength of large cardinals carried by the model.

Uniformity of degree structures

Note that clause (1) holds in all the four types of models we've discussed: L, $L[\mu_0, \ldots, \mu_n]$, $L[\bar{\mu}]$, the canonical model in Yang Sen's theorem.

Remark.

Among (fine structure) inner models, the "richness" of the degree structures seems to correspond to the location of λ in the inner model, rather than to the strength of the inner model.

However the main tool in our analysis are Covering Lemmas, which hold only up to a Mitchell's model for certain sequences of measures. Beyond that point, only weak covering properties are known.

A Conjecture

Conjecture (Woodin)

In Jensen-Steel model (2013), at every singular λ with $\mathrm{cf}(\lambda) = \omega$, the generalized degrees at λ are well ordered above some degree.

Test question

In Jensen-Steel model (2013), the generalized degrees at $\aleph_\omega$ are well ordered above some degree.

What makes even the test question challenging is that unlike in $L[\bar{\mu}]$-type models, in the construction of $L[E]$, E is an extender sequence, partial extenders may be added before it reaches $\aleph_\omega$.

Answer to the test question

Mitchell-Schimmerling's recent work: *Covering at limit cardinal of* K (2019), provides us the missing tool to answer both questions.

Theorem (Schindler-S.)

Assume there is no transitive model of ZF^{-}[a] *has an inner model with a Woodin cardinal. Let* $L[E]$ *be a fully iterable pure extender model. Then the* ZF^{-}*-degrees at* $\aleph_\omega$ *are well ordered.*

[a] Here ZF^{-} is a sufficient rich fragment of ZF which allows us to carry out relevant proofs in the core model theory.

Answer to the conjecture

Theorem (Schindler-S.)

Assume there is no transitive model of ZF$^-$ *has an inner model with a Woodin cardinal. Let $L[E]$ be a fully iterable pure extender model. Let λ be a singular cardinal of $L[E]$.*

1. *If λ is not an ω-limit of measurable cardinals, then*

   $$L[E] \models \text{“ZF}^-\text{-degrees at } \lambda \text{ are well ordered above some degree”}.$$

2. *If λ is an ω-limit of measurable cardinals, then*

   $$L[E] \models \text{“There are incomparable ZF}^-\text{-degrees at } \lambda\text{”}.$$

Next question

What is the criterion for finding minimal covers in a fully iterable pure extender model?

Thank you!