pp-elimination and stability in a continuous logic environment

Fudan conference on Model Theory and Philosophy of Mathematics

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- We want to do something analogous in a suitable "continuous logic" environment.
- Some of the inspiration or motivation comes from Hrushovski's recent work on simplicity of the theory of finite fields equipped with an additive character in a continuous logic environment (as well as our asking the question what, if anything, is the continuous analogue of a 1-based group).

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- However this will be stability in a suitable version of continuous logic, which will be described in the next section.

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- When $\mathcal{C}$ is a metric space, we could treat this set up with the formalism of BY-B-H-U, by viewing $(M^-, f, \mathcal{C})$ as a 2-sorted structure, with the metric d on $\mathcal{C}$, as well as all continuous functions from $\mathcal{C}$ to $[0,1]$ as real valued relations on $\mathcal{C}$. (As Henson pointed out.)

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- But it is convenient and conceptually simpler (for me at least), to choose an essentially equivalent formalism, which is closer to the Henson-Iovino positive bounded logic of normed vector spaces, as well as so-called "positive logic".

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- Let L be the 2-sorted language, with a sort for M^- equipped with all its L^--structure, as well as a sort for $\mathcal{C}$, a symbol for the function f, and predicates for all closed subsets of the various Cartesian powers of $\mathcal{C}$ (a bit of overkill, but never mind).

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- I guess we could also call this a $\mathcal{C}$-L-structure (in analogy with ω-models in second order arithmetic).

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- So the set of CL L-formulas is just a subset of the set of all L-formulas.
- A CL-sentence is a CL-formula without free variables.
- So we have a class of structures, the CL-structures, a class of formulas, the CL-formulas, together with the satisfaction relation, induced from the L-structures and L-formulas.

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- If N is a CL-structure, $\phi(\bar{x})$ a CL-formula and $\bar{a}$ a tuple from N, then we write $N \models_{approx} \phi(\bar{a})$ if $N \models \psi(\bar{a})$ for each approximation ψ to ϕ.

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- In saturated CL-structures (our "universal domains") which are the right places to work, approximate truth coincides with truth.
- The analogue of a complete theory is the approximate CL-theory of some CL-structure, equivalently the CL-theory of some (ω)-saturated CL-structure.

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- Here the type of a tuple $\bar{a}$ is the set of CL-formulas true of $\bar{a}$ in M.
- This agrees with the notion of stability for classical first order theories, as well as in continuous logic in the sense of BY-B-H-U.

Remarks and questions

- As an example, our main theorem implies that the structure $(\mathbb{R}, +, -, 0)$ equipped with the canonical covering map $f : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is stable (i.e, its approximate CL-theory is stable).

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- We could look at a twist of this where the domain is equuipped with the full field structure and the target is just viewed as a compact space.
- There has been quite a bit of work around describing stable expansions of $(\mathbb{Z},+)$. We could more generally ask about CL-stable "expansions" of $(\mathbb{Z},+)$ by a map $f:\mathbb{Z}$ to $[0,1]$ or to any compact space.

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- where ϕ is a finite conjunction of atomic L^--formulas, I, J are (possibly empty) subsets of the index sets of $\bar{x}$, $\bar{y}$ respectively, and c_i, d_j are in $\mathbb{T}$ for $i \in I$, $j \in J$.

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Lemma 0.1

If $\bar{a}$ and $\bar{b}$ are n-tuples from A with the same pp^* type (namely they satisfy in M exactly the same pp^*-formulas), then $\bar{a}$, $\bar{b}$ have the same type in M (i.e. satisfy the same CL-formulas).

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- The key point is to express the emptyness of the intersection of a pp^*-formula (with parameters) with a finite conjunction of negated pp-formulas (with parameters) by a pp^*-formula and negated pp-formulas (in the parameters).
- It is done using the Neumann lemma and inclusion-exclusion principle, elaborating on the classical proof of pp-elimination for modules.

Final remarks

- Tran and Walsberg show that if $f : \mathbb{Z} \to S_1$ is given by an "irrational rotation", then adjoining to $(\mathbb{Z}, +)$ the preimage of a small interval around the identity in S_1, gives a dp-minimal proper expansion, which is hence unstable (by a result of Conant and me).

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- Note that if G is a saturated stable group (as a first order structure), and we add a new sort for the compact group G/G^0, then the resulting CL-structure is also stable.
- But our results show that not all CL-stable structres $(G, f, \mathcal{C})$ with f a homomorphism arise this way.