Effective Definability of Kolchin Polynomials

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August 21, 2021

Fudan Model Theory and Philosophy of Mathematics Conference

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Outline

- Problem and the main result
- Main ingredients of the proof
- Some applications

Notation

- Let (K, Δ) be a differentially closed field of characteristic zero with $\Delta = \{\delta_1, \ldots, \delta_m\}$ a set of distinguished commuting derivations. Assume K is a saturated model of $\mathrm{DCF}_{0,m}$.
- Let $\mathbf{y} = (y_1, \ldots, y_n)$ be an n-tuple of Δ-indeterminates. For $\xi \in \mathbb{N}^m$, denote $\delta^\xi = \delta_1^{\xi_1} \cdots \delta_m^{\xi_m}$ and $\mathrm{ord}(\xi) = \sum_{i=1}^m \xi_i$. The Δ-polynomial ring in $\mathbf{y}$ over $k (\subseteq K)$ is

  $$k\{\mathbf{y}\} := k[\delta^\xi(y_j) : \xi \in \mathbb{N}^m, j = 1, \ldots, n].$$

  The order of $f \in k\{\mathbf{y}\}$ is $\max\{\mathbf{ord}(\xi) : \delta^\xi(y_j) \text{ occurs in } f\}$.
- A Δ-**variety** $V \subset K^n$ over k is the set of differential zeros of some $F \subset k\{\mathbf{y}\}$, i.e., $V = \{\mathbf{a} \in K^n : f(\mathbf{a}) = 0, \forall f \in F\}$.
- A Δ-variety V is irreducible if it is not the union of two proper Δ-subvarieties. An irreducible Δ-variety has a generic zero.

Kolchin polynomials

- A polynomial $p \in \mathbb{Q}[t]$ is called a **numerical polynomial** if $p(s) \in \mathbb{Z}$ for all integers s. It can always be written as

$$p(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i}, \quad \text{where } a_i \in \mathbb{Z}.$$

- Let $V \subset \mathbb{A}^n$ be an irreducible Δ-variety over k with a generic zero $\mathbf{a}$. Kolchin proved that there is a numerical polynomial $\omega_V(t)$ s.t. for all sufficiently large $s \in \mathbb{N}$,

$$\omega_V(s) = \text{tr.deg}\, k(\delta^{\xi}\mathbf{a} : \text{ord}(\xi) \leq s)/k.$$

Call $\omega_V(t)$ the **Kolchin polynomial** of V.

Examples

- If V is defined by pure algebraic polynomials (i.e., differential polynomials of order 0), then $\omega_V(t) = d$, the dimension of V.

- In the ordinary differential case (i.e., $m = 1$):

$$\omega_V(t) = d(t+1) + s,$$

  Here d is the **differential dimension** of V and s **order** of V.

  Example. Let $V = \mathbb{V}(\delta^2(y_1) + y_1 y_2) \subset K^2$. Then $\omega_V(t) = (t+1) + 2$.

- Let $m = 2$, $n = 3$ and $V = \mathbb{V}(\delta_1(y_1), y_2 - \delta_1(y_1), y_3 - \delta_2(y_1))$. Then $\omega_V(t) = (t+1) + 1$

Properties of Kolchin Polynomials

The Kolchin polynomial has the following important properties:

- $\mathbf{deg}\,\omega_V(t) \leq m\,(= \text{card}(\Delta))$, hence $\omega_V(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i}$. The coefficient a_m is equal to the Δ-dimension of V.
- The collection of Kolchin polynomials is well-ordered by eventual domination (i.e., $p(t) \leq q(t)$, if $p(s) \leq q(s)$ for all sufficiently large $s \in \mathbb{N}$).
- If $V \subseteq W$ are irreducible Δ-varieties over k, then $\omega_V(t) \leq \omega_W(t)$, and equality holds iff $V = W$.

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Given an arbitrary Δ-variety V, the **Kolchin polynomial** of V is defined as

$$\omega_V(t) := \max_{\leq}\{\omega_W(t) : W \text{ is an irreducible component of } V\}.$$

Main problem

Let $F(\mathbf{x},\mathbf{y}) \subseteq k\{\mathbf{x},\mathbf{y}\}$ be a collection of Δ-polynomials. For each $\mathbf{a} \in K^{|\mathbf{x}|}$, $F(\mathbf{a},\mathbf{y}) \subseteq k\langle\mathbf{a}\rangle\{\mathbf{y}\}$ defines a Δ-variety $V_{\mathbf{a}}$ over $k\langle\mathbf{a}\rangle$. Any such collection of $V_{\mathbf{a}}$'s is called a **definable family of Δ-varieties**, denoted by $(V_{\mathbf{a}})$.

Main Problem: Given a definable family of Δ-varieties $(V_{\mathbf{a}})$ and a numerical polynomial $p(t)$, is the set

$$\{\mathbf{a} : \omega_{V_{\mathbf{a}}}(t) = p(t)\}$$

definable in the structure (K,Δ)? Can one prove this effectively?

Definability in Algebraic and Ordinary Differential Cases

- **Algebraic Case:**
  Relative to ACF_0, the dimension is definable in families:
  Let $V_{\bar{a}} = \mathbb{V}\big(F(\bar{a}, y_1, \ldots, y_n)\big)$. For each $d \in \mathbb{N}$,
  $\{\bar{a} : \mathbf{dim}(V_{\bar{a}}) = d\}$ is a constructible set.

- **Ordinary Differential Case (i.e., $m = 1$):**
  Relative to DCF_0, the δ-dimension, order and the Kolchin polynomial are definable in families (Freitag et al., 2017):
  Let $V_{\bar{a}} = \mathbb{V}\big(F(\bar{a}, y_1, \ldots, y_n)\big)$. For each $d, h \in \mathbb{N}$,
  $\{\bar{a} : \omega_{V_{\bar{a}}}(t) = d(t+1) + h\}$ is δ-constructible.

Main Result: Definability of Kolchin Polynomials

The answer to the main problem is **yes**.

Theorem (Freitag, León Sánchez, Li, 2020) Let $(V_{\mathbf{a}})$ be a definable family of Δ-varieties and $p(t)$ a numerical polynomial. Then the sets

$$\{\mathbf{a} : \omega_{V_{\mathbf{a}}}(t) \geq p(t)\}$$

and

$$\{\mathbf{a} : \omega_{V_{\mathbf{a}}}(t) = p(t)\}$$

are definable in the structure (K, Δ).

Furthermore, for a fixed $\ell \in \mathbb{N}$,

$$\{\mathbf{a} : \omega_{V_{\mathbf{a}}}(t) = p(t) \text{ and } V_{\mathbf{a}} \text{ has } \ell \text{ components with Kol-poly } p(t)\}$$

is also definable in the structure (K, Δ).

Main Ingredients of the Proof

Main Idea

$$\begin{array}{rc} \text{Differential:} & (V_{\mathbf{a}}) \\ & \downarrow \text{Prolongation} \\ \text{Algebraic:} & (B_s(V_{\mathbf{a}})) \end{array}$$

Key Points.

- To show $(B_s(V_{\mathbf{a}}))$ is a definable family of algebraic varieties;
- Derive bounds s_0, s_1 such that for components W_i of $V_{\mathbf{a}}$,
  - $\omega_{W_i}(s) = \dim(B_s(W_i))$ for all $s \geq s_0$;
    (Bound for Hilbert-Kolchin regularity)
  - $\omega_{W_1} > \omega_{W_2}$ iff $\omega_{W_1}(s) > \omega_{W_2}(s)$ for all $s > s_1$.
    (Bound witnessing eventual domination)

Prolongation of Differential Varieties

Given $\mathbf{b} \in K^n$ and $s \in \mathbb{N}$, let $\nabla_s(\mathbf{b})$ be the tuple in $K^{n \cdot \binom{m+s}{m}}$ consisting of $\mathbf{b}$ and its derivatives $\delta^{\xi}(\mathbf{b})$ of order at most s.

Def. Given a Δ-variety $V \subset K^n$, the prolongation of V is defined as

$$B_s(V) := \nabla_s(V)^{\text{Z-cl}} \subseteq K^{n \cdot \binom{m+s}{m}}$$

where $*^{\text{Z-cl}}$ denotes Zariski-closure.

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Proposition. Let $(V_{\mathbf{a}})$ be a definable family of Δ-varieties, and let $s \in \mathbb{N}$. Then $(B_s(V_{\mathbf{a}}))$ has the structure of a definable family, and one can effectively compute a formula defining this family.

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Proof. Use the fact that Zariski-closure being definable in families of differential varieties following from results by Freitag and León Sánchez (2016).

Definability for Prolongation of Differential Varieties

Algebraic Fact. (Hrushovski(1992), Johnson(2017)) Let $(X_{\mathbf{a}})$ be a definable family of algebraic varieties. For fixed d and ℓ, the set

$$\{\mathbf{a} : \mathbf{dim}\, X_{\mathbf{a}} \geq d \text{ and } X_{\mathbf{a}} \text{ has at least } \ell\text{-many top-dim components}\}$$

is definable by a formula in the language of rings. Moreover, a formula defining this set can be effectively computed.

Proposition. Let (V_a) be a definable family of Δ-varieties, and $s, d, \ell \in \mathbb{N}$. Then the set

$$\{\mathbf{a} : \mathbf{dim}\, B_s(V_{\mathbf{a}}) \geq d \text{ and has at least } \ell\text{-many top-dim components}\}$$

is definable, and a formula can be effectively computed.

Hilbert-Kolchin regularity

For which values s do we get

$$\omega_V(s) = \text{tr.deg}\, k(\delta^\xi \mathbf{a} : \text{ord}(\xi) \leq s)/k\ ?$$

The smallest r_0 such that the above holds for all $s \geq r_0$ is called the **Hilbert-Kolchin regularity** of V.

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For $r \geq 0$ and $n, m \geq 1$, we build $C^n_{r,m}$ recursively as follows:

$$C^1_{0,m} = 0, \;\; C^1_{r,m} = A(m-1, C^1_{r-1,m}), \;\; \text{and} \;\; C^n_{r,m} = C^1_{C^{n-1}_{r,m},m},$$

where $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the Ackermann function.

For example, $C^n_{r,1} = r$, $C^n_{r,2} = 2^n r$ and $C^1_{r,3} = 3(2^r - 1)$.

Bound for Hilbert-Kolchin regularity

Theorem. Let $V \subseteq K^n$ be a Δ-variety over k (not necessarily irreducible) defined by Δ-polynomials of order at most r. Set

$$s_0 := m\, C^n_{r,m} \binom{C^n_{r,m} + m - 1}{C^n_{r,m}} - m.$$

Then, for all $s \geq s_0$ and any irreducible component W of V,

$$\omega_W(s) = \mathrm{trdeg}_k\, k(\delta^\xi b_i : \mathbf{ord}\,\xi \leq s, i = 1, \ldots, n)$$

where $b = (b_1, \ldots, b_n)$ is a generic point of W.

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Remark. (1). In ordinary diff case, $s_0 = r - 1$, so Hilbert-Kolchin regularity of V is bounded by $r - 1$. This generalizes the result of D'Alfonso et al. (2009) for quasi-regular ordinary diff systems.

(2). The proof uses an upper bound for the order of **characteristic sets** obtained by Gustavson and León Sánchez (2018).

Bound for eventual domination

Theorem. Let $V \subseteq K^n$ be a Δ-variety over k defined by Δ-polynomials of order at most r. Set

$$s_1 := n\,2^{m+1}\,m!\,D^m + 1$$

where

$$D = C^n_{r,m}\binom{C^n_{r,m} + m - 1}{C^n_{r,m}}.$$

Suppose W_1 and W_2 are components of V. Then, $\omega_{W_1} > \omega_{W_2}$ if and only if $\omega_{W_1}(s) > \omega_{W_2}(s)$ for all $s > s_1$.

Applications to weak irreducibility

A Δ-variety V is called **weakly irreducible** if it has exactly **one component** of maximal Kolchin polynomial (though it might have several components).

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Corollary. Given a definable family $(V_{\mathbf{a}})$ of Δ-varieties, the set

$$\{\mathbf{a} : V_{\mathbf{a}} \text{ is weakly irreducible}\}$$

is definable in the structure (K, Δ).

Remark. (Irreducibility vs Weak Irreducibility)
If we remove the word "weakly" in the corollary, the statement is still open. It is equivalent to the generalized Ritt problem, a long-lasting open problem in differential algebra since 1950s.

Applications to finiteness of Kolchin polynomials

A definable family admits only finitely many Kolchin polynomials:

Corollary. Let $(V_{\mathbf{a}})$ be a definable family of Δ-varieties. Then the set

$$\{\omega_{V_{\mathbf{a}}}: \text{ as } V_{\mathbf{a}} \text{ varies in the family}\}$$

is finite.

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The degree of the Kolchin polynomial of V is called the Δ-type of V, denoted by τ_V.

Corollary. Let $(V_{\mathbf{a}})$ be a definable family of Δ-varieties and $d \in \mathbb{N}$. Then the set

$$\{\mathbf{a} : \tau(V_{\mathbf{a}}) = d\}$$

is definable.

Thanks!