Definable sets in valued ω-free PAC fields

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§1 The Main Theorem

The Main Theorem

Theorem(H.)

Suppose that $(K, V) \models \mathrm{PACVF}^{\omega}_{p^e}$ for a natural number $e < \infty$ and that W is a valuation on K^{alg} extending V. Then every $\mathscr{L}_{\mathrm{div}}$-definable set in K is W-dense in an $\mathscr{L}_{\mathrm{div}}$-definable set in K^{alg} with parameters from K. More precisely, if $\varphi(\boldsymbol{x}, \boldsymbol{a})$ is an $\mathscr{L}_{\mathrm{div}}$-formula with $\boldsymbol{a} \in K^n$, then there exists an $\mathscr{L}_{\mathrm{div}}$-formula $\tilde{\varphi}(\boldsymbol{x}, \boldsymbol{b})$ with $\boldsymbol{b} \in K^m$ such that $\varphi(K)$ is a W-dense subset of $\tilde{\varphi}\left(K^{\mathrm{alg}}\right)$.

Convention

Definable: always presumes parameters from the underlying universe.

§2 $\mathrm{PACVF}_{p^e}^{\omega}$

PAC Fields

Definition

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Remark

A field is PAC iff it is existentially closed in every regular field extension.

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Definition

The **language of rings** is the first-order language

$$\mathscr{L}_{\mathrm{r}} := \{+, -, \times, 0, 1\}.$$

Fact

Being PAC is a first-order property in $\mathscr{L}_{\mathrm{r}}$.

Valued Fields

Definition

A field K with a distinguished subring V satisfying

$$(\forall x \in K^\times)\left[(x \in V) \bigvee \left(x^{-1} \in V\right)\right]$$

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Specifying a valuation ring V is the same as specifying a **valuation** map

$$v : K \twoheadrightarrow \Gamma \cup \{\infty\}$$

satisfying

(1). $v(x) = \infty$ iff $x = 0$,

(2). $v(xy) = v(x) + v(y)$,

(3). $v(x + y) \geq \min\{v(x), v(y)\}$,

where Γ is an ordered abelian group.

Definition

Suppose that (K, V) is a valued field. Then the topology generated by basic open sets of the form

$$\{x \in K \mid v(x-a) > \gamma\}, \quad a \in K, \gamma \in \Gamma,$$

makes K a topological field. The topology is called the **topology induced by** V.

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So, W**-dense** is short for dense with respect to the topology induced by the valuation ring W.

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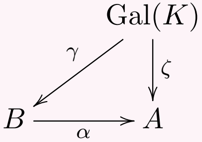
$\mathscr{L}_{\text{div}} := \mathscr{L}_{\text{r}} \cup \{|\}$ denotes the **language of valued fields** (or valued rings). On a valued field, the division predicate is interpreted as

$$x \mid y \Leftrightarrow (\exists z \in V)(y = xz).$$

ω-free Fields

Definition

A field K is **ω-free** if every finite embedding problem for $\mathrm{Gal}(K)$ is solvable, that is, given any two epimorphisms of profinite groups $\zeta : \mathrm{Gal}(K) \to A$ and $\alpha : B \to A$, where B is a finite group, there always exists an epimorphism $\gamma : \mathrm{Gal}(K) \to B$ such that the following diagram commutes.



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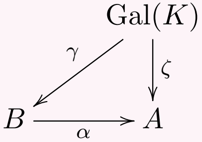

Fact

Being ω-free is a first order property in $\mathscr{L}_{\mathrm{r}}$.

Exponent of Imperfectness

Fact-definition

Suppose that K is a field with $\mathrm{char}(K) = p > 0$. Then there exists a unique $e \in \omega \cup \{\infty\}$ such that $[K : K^p] = p^e$. This e is called the **exponent of imperfecness** of K and $[K : K^p]$ the **degree of imperfectness**.

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A set of elements $b_0, \ldots, b_{n-1} \in K$ are p**-independent** in K if all the p-monomials in $b_0, \ldots, b_{n-1}$ of the form

$$\prod_{j \in n} b_j^{i(j)}, \quad i : n \to p$$

are linearly independent over K^p.

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A set of elements $b_0, \ldots, b_{n-1} \in K$ are p-**independent** in K if all the p-monomials in $b_0, \ldots, b_{n-1}$ of the form

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A p-**basis** is a set of maximally p-independent elements in K.

$[K : K^p] = p^e$ iff K has a p-basis with exactly e elements.

Definition

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A field extension L/K is **separable** if the p-independence relation is preserved. If furthermore, K is relatively algebraically closed in L, then L/K is said to be **regular**.

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§3 Quantifier Elimination

Relative p-coordinate Function Symbols

Definition(Srour, 1986)

Suppose that K is a field of positive characteristic p. When $n > 0$ and $i \in p^n$, we define the **relative p-coordinate functions** $\lambda_{n,i}(x; y_1, \ldots, y_n)$ as follows:

(1). if $y_1, \ldots, y_n$ are not p-independent in K or $x \notin K^p(y_1, \ldots, y_n)$, then $\lambda_{n,i}(x; y_1, \ldots, y_n) = 0$;

(2). otherwise $\lambda_{n,i}(x; y_1, \ldots, y_n)$ is the p-th root of the unique i-th coefficient of x with respect to $y_1, \ldots, y_n$ when x is written as a linear combination of p-monomials in $y_1, \ldots, y_n$ over K^p.

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If $n = 0$, then we defined $\lambda_{n,i}(x) = \lambda_{0,0}(x)$ to be $x^{1/p}$ if $x \in K^p$, 0 if $x \notin K^p$.

Maximal Splitting Coefficient Function Symbols

Definition(H.)

Let K be any field. Given a polynomial $f(X) \in K[X]$ with $\deg(f) > 0$, we define the **maximal splitting factor** of $f(X)$ to be the unique monic polynomial $g(X)$ satisfying the following two conditions:

(1). $g(X)$ splits into linear factors over K or $g(X) = 1$;

(2). there exists an polynomial $h(X) \in K[X]$ without roots in K such that $f(X) = g(X)h(X)$.

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Definition(H.)

Given $n+1$ elements $a_0, \ldots, a_{n-1}, a_n$ in a field K, denote that the maximal splitting factor of

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$$

over K by

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where $b_n, \ldots, b_0 \in K$, some of which could be zeros.

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where $b_n, \ldots, b_0 \in K$, some of which could be zeros. For each $0 \leq i \leq n$, we define the i-th **splitting coefficient** to be

$$\theta_{n,i}(a_0, a_1, \ldots, a_n) := b_i.$$

Notation

$\mathscr{L}_{\theta,p,\mathrm{div}} := \mathscr{L}_{\mathrm{div}} \cup \{\lambda_{n,i}\}_{n,i} \cup \{\theta_{m,j}\}_{m,j}.$

$\mathrm{PACVF}^{\omega}_{p^e}$ has QE

Theorem(H.)

For any $e \in \omega \cup \{\infty\}$, the theory of ω-free pseudo-algebraically closed non-trivially valued fields of characteristic p with exponent of imperfectness e in $\mathscr{L}_{\theta,p,\mathrm{div}}$, denoted by $\mathrm{PACVF}^{\omega}_{p^e}$, admits quantifier elimination.

§4 Outline of The Proof

Recall The Main Theorem

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We may assume that φ is of the form:

$$\bigvee_{\alpha=1}^{\beta}\Big\{[\wedge_{i=1}^{n}(f_{\alpha,i}(\boldsymbol{x},\boldsymbol{a})=F_{\alpha,i}(\boldsymbol{x},\boldsymbol{a}))]\bigwedge[\wedge_{j=1}^{m}(g_{\alpha,j}(\boldsymbol{x},\boldsymbol{a})\neq 0)]\bigwedge \bigwedge[\wedge_{k=1}^{s}(x_{\alpha,i_k}\mid x_{\alpha,j_k})]\bigwedge[\wedge_{l=1}^{t}(x_{\alpha,u_l}\nmid x_{\alpha,v_l})]\Big\},$$

where for all α,i,j,k,l, $f_{\alpha,i}$, $F_{\alpha,i}$ and $g_{\alpha,j}$ are $\mathscr{L}_{\theta,p,\mathrm{div}}$-terms, $x_{\alpha,i_k},x_{\alpha,j_k},x_{\alpha,u_l}$ and x_{α,v_l} are entries of $\boldsymbol{x}$. In fact, for each α and i, we may assume that $f_{\alpha,i}(\boldsymbol{x},\boldsymbol{a})=F_{\alpha,i}(\boldsymbol{x},\boldsymbol{a})$ is one of the three forms:

(1). $f_{\alpha,i}$ is a polynomial over $\mathbf{Z}[\boldsymbol{a}]$ with variables in $\boldsymbol{x}$, $F_{\alpha,i}=0$;

(2). $f_{\alpha,i}$ is a relative p-coordinate function with arguments a subtuple of $\boldsymbol{x}$, $F_{\alpha,i}$ is an entry of $\boldsymbol{x}$;

(3). $f_{\alpha,i}$ is a splitting-coefficient function $\theta_{\nu,\mu}$ for some numbers ν and μ with arguments a subtuple of $\boldsymbol{x}$, $F_{\alpha,i}$ is an entry of $\boldsymbol{x}$.

Meanwhile, for each α and j, $g_{\alpha,j}$ is a polynomial over $\mathbf{Z}[\boldsymbol{a}]$ with variables in $\boldsymbol{x}$.

We may assume that φ is of the form:

$$\bigvee_{\alpha=1}^{\beta} \Big\{ [\wedge_{i=1}^{n} (f_{\alpha,i}(\boldsymbol{x},\boldsymbol{a}) = F_{\alpha,i}(\boldsymbol{x},\boldsymbol{a}))] \bigwedge [\wedge_{j=1}^{m} (g_{\alpha,j}(\boldsymbol{x},\boldsymbol{a}) \neq 0)] \bigwedge$$

$$\bigwedge [\wedge_{k=1}^{s} (x_{\alpha,i_k} \mid x_{\alpha,j_k})] \bigwedge [\wedge_{l=1}^{t} (x_{\alpha,u_l} \nmid x_{\alpha,v_l})] \Big\},$$

where for all α, i, j, k, l, $f_{\alpha,i}$, $F_{\alpha,i}$ and $g_{\alpha,j}$ are $\mathscr{L}_{\theta,p,\mathrm{div}}$-terms, $x_{\alpha,i_k}, x_{\alpha,j_k}, x_{\alpha,u_l}$ and x_{α,v_l} are entries of $\boldsymbol{x}$. In fact, for each α and i, we may assume that $f_{\alpha,i}(\boldsymbol{x},\boldsymbol{a}) = F_{\alpha,i}(\boldsymbol{x},\boldsymbol{a})$ is one of the three forms:

(1). $f_{\alpha,i}$ is a polynomial over $\mathbf{Z}[\boldsymbol{a}]$ with variables in $\boldsymbol{x}$, $F_{\alpha,i} = 0$;

(2). every realization of formula has coordinates $x_{\alpha,i_k}, x_{\alpha,j_k}, x_{\alpha,u_l}, x_{\alpha,v_l}$ non-zero for all k and l;

(3). $f_{\alpha,i}$ is a splitting-coefficient function $\theta_{\nu,\mu}$ for some numbers ν and μ with arguments a subtuple of $\boldsymbol{x}$, $F_{\alpha,i}$ is an entry of $\boldsymbol{x}$.

Meanwhile, for each α and j, $g_{\alpha,j}$ is a polynomial over $\mathbf{Z}[\boldsymbol{a}]$ with variables in $\boldsymbol{x}$.

The realization set of the formula $\theta_{\nu,\mu}(y_{e_0}, y_{e_1}, \ldots, y_{e_\nu}) = y_{e_{\nu+1}}$ in a field is contained in the realization set of the following existential formula

$$\bigvee_{j=0}^{\nu} \Bigg[(\exists w_0, \ldots, w_\nu)(\exists z_1, \ldots, z_j) \left(\text{“}y_{e_{v+1}} \text{ is the } \mu\text{-th coeff. of } \prod_{i=0}^{j} (X - z_i)\text{”} \right) \wedge \text{“} \sum_{i=0}^{\nu} x_i X^i \text{ equals the product of } \prod_{i=0}^{j} (X - z_i) \text{ and } \sum_{i=0}^{\nu-j} w_i X^i\text{”} \Bigg].$$

Note that by asserting that in the formula above each of the $\sum_{i=0}^{\nu-j} w_i X^i$ does not have a root, the above formula is equivalent to $\theta_{\nu,\mu}(y_{e_0}, y_{e_1}, \ldots, y_{e_\nu}) = y_{e_{\nu+1}}$.

Fact(from Poonen's book on rational points, 2017)

Let X be a finite-type scheme over a field k such that $X(k)$ is Zariski-dense in X, if X is integral then X is geometrically integral.

Kollár's Density Theorem(Kollár, 2007)

Suppose that K is a PAC field, V a non-trivial valuation on K^{alg}. Then for any geometrically integral K-variety X, $X(K)$ is V-dense in $X(K^{\mathrm{alg}})$.

Theorem(Bary-Soroker, 2012)

Let K be a PAC field, Y an absolutely irreducible smooth K-variety with ring of regular functions R, $f(X) \in R[X]$ a separable monic polynomial, and P a partition of $\deg(f)$. Assume that the induced embedding problem has a solution whose orbit type is P. Then there exists a Zariski dense set of $\mathfrak{p} \in Y(K)$ such that $\varphi_{\mathfrak{p}}(f)$ is a separable polynomial of factorization type P.

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The induced embedding problem is the diagram

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The induced embedding problem is the diagram

$$\begin{array}{ccc} & & \mathrm{Gal}(K) \\ & \overset{\theta}{\swarrow} & \downarrow \pi \\ \mathrm{Gal}(F/E) & \xrightarrow[\alpha]{} & \mathrm{Gal}(L/K). \end{array}$$

It is said to have a (weak) solution if there exists a continuous homomorphism $\theta : \mathrm{Gal}(K) \to \mathrm{Gal}(F/E)$ such that the above diagram added with θ commutes.

Corollary(H.)

Let K be a pseudo-algebraically closed field, W a valuation ring on K^{alg}, Y a geometrically integral smooth K-variety with its ring of regular functions R, $f(X) \in R[X]$ a separable monic polynomial, P a partition of $\deg f$. Assume that the induced embedding problem has a solution whose orbit type is P. Then there exists a subset of $\mathfrak{p} \in Y(K)$ W-dense in $Y(K^{\text{alg}})$ such that $\varphi_{\mathfrak{p}}(f)$ is a separable polynomial of factorization type P.

Thank you!