The Tennenbaum phenomenon for computable quotient presentations of models of arithmetic and set theory

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Fudan University, Shanghai
21-24 August 2021

This is joint work with Michał Tomasz Godziszewski, a fun little project.

[GH17] M. T. Godziszewski and J. D. Hamkins, “Computable Quotient Presentations of Models of Arithmetic and Set Theory,” in Proceedings WoLLIC 2017, Springer, 2017, p. 140–152. 10.1007/978-3-662-55386-2 10, arXiv:1702.08350, blog post.

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There answer is no, there is no such model.

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The set of standard n in A_t^M is a computable separation of A and B, contradiction. □

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Again the answer is no.

And there is no nonstandardness requirement for the set theory case.

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So C provides a computable separation, contradiction. □

Generalizing Tennenbaum

We are interested in various generalizations of the Tennenbaum phenomenon.

Computable quotient presentations

A *computable quotient presentation* of a structure $\mathcal{A}$ is a computable structure $\langle \mathbb{N}, \star, *, \cdots \rangle$ together with an equivalence relation E such that

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In a language with relations, it is natural to consider *computably enumerable* quotient presentations, which allow the pre-quotient relation to be merely c.e.

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He proposes them as a fruitful alternative approach to computable model theory.

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Distinct numbers n, m represent distinct elements of the structure.

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Part of the attraction of quotient presentations is to loosen the computational grip on the identity relation.

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1. No nonstandard model of arithmetic admits a computable quotient presentation by a computably enumerable equivalence relation on the natural numbers.
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Some variations remains open.

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Slogan: the Henkin model is a quotient of the term algegra. □

Fundamental fact of universal algebra

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For this reason, quotient presentations are often especially relevant for universal algebra, that is, in purely functional languages.

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For which equivalence relations E can a structure have a computable presentation on domain $\mathbb{N}/E$?

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Problems arrive when the language has relation symbols.

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There is no computable structure $\langle \mathbb{N}, \oplus, \odot \rangle$ and a c.e. relation E for which $\langle \mathbb{N}, \oplus, \odot \rangle / E$ is a nonstandard model of arithmetic.

Proof

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This is a computable separation of A and B, contradiction. □

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That is, we only require $\oplus$ to be computable, not both $\odot$ and $\oplus$.

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It follows that every standard system must have some non-c.e. sets.

But our argument above showed that if E is c.e., then every set in the standard system is c.e., contradiction.

Full language of arithmetic

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We shall also consider the language with the reflexive order $\{+, \cdot, 0, 1, \leq\}$.

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Lemma

Suppose that E is an equivalence relation on the natural numbers.

1. *If $\lhd$ is computable and $\langle \mathbb{N}, \lhd \rangle / E$ is a strict linear order, then E is computable.*

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By the lemma, E is also co-c.e., hence computable. Now pick representatives, violate Tennenbaum. □

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Alternative: ignore $<$ and apply earlier theorem.

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Suppose $\langle \mathbb{N}, \oplus, \odot, \trianglelefteq \rangle / E$ works.

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The conjecture: some nonstandard model of arithmetic has computable quotient presentation with co-c.e. *E*.

Let us refute this for PA in the language $\{+, \cdot, \leq\}$, even for c.e. quotient presentations.

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No nonstandard model of arithmetic in $\{+, \cdot, \leq\}$ has a c.e. quotient presentation by any equivalence relation, of any complexity.

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So this reduces to the first theorem. □

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Theorem (Godziszewski, Hamkins)

No nonstandard model of true arithmetic has a computable quotient presentation $\langle \mathbb{N}, \oplus, \odot \rangle$ using a co-c.e. equivalence relation E.

First proof

Suppose $M = \langle \mathbb{N}, \oplus, \odot \rangle / E$ is nonstandard model of true arithmetic, where E is a co-c.e. equivalence relation.

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This separates, and is Δ_2, hence computable from $0'$, contradiction.

Alternative proof

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This violates Scott's theorem that the standard system will have sets of higher complexity.

Sharper result

Theorem (Godziszewski, Hamkins)

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The proof we gave shows it is enough if merely $0'$ is in the standard system.

This is weaker than Σ_1-sound, since a simple compactness argument enables us to insert any particular set into standard system.

Another variation

Corollary

No nonstandard model of arithmetic in the language $\{+, \cdot, 0, 1, <\}$ *and with* $0'$ *in its standard system has a computably enumerable quotient presentation by any equivalence relation, of any complexity.*

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Proof.

If $\langle \mathbb{N},\oplus,\odot,\bar{0},\bar{1},\lhd\rangle/E$ is c.e. quotient presentation, then by the lemma, E must be co-c.e., and so this is ruled out by previous theorem. □

Central case still open

Question

Is there a nonstandard model of PA in language $\{+, \cdot, 0, 1, <\}$ with a computably enumerable quotient presentation by some co-c.e. equivalence relation? Equivalently, is there a nonstandard model of PA in that language with a computably enumerable quotient presentation by any equivalence relation, of any complexity?

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Khoussainov conjectured positive answer; we expect a negative answer.

Set theory

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No model of ZFC *has a computable quotient presentation, by an equivalence relation of any complexity.*

That is, there is no computable relation ϵ and equivalence relation E, of any complexity, such that $\langle \mathbb{N}, \epsilon \rangle / E$ is a model of ZFC.

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It follows that every set in standard system of $\langle \mathbb{N}, \epsilon \rangle / E$ is computable, contradiction. $\square$

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Now argue from that case. . .

No c.e. quotients by co-c.e. E

Theorem (Godziszewski, Hamkins)

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This is enough to run the computable inseparability arguments. □

Co-c.e. E on a relational structure

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Conclusion

Every computable quotient presentation of a relational structure by a co-c.e. relation has a computable presentation.

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Proved by using S, Sing, Doub etc. below a nonstandard number.

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Can again get *E* decidable for members of any given set, by using Sing.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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Oxford University

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