# Definable Fields in Various Dp-minimal Fields

## 2021 Fudan Model Theory and Philosophy of Mathematics Conference

Yatir Halevi

Fields Institute for Research in Mathematical Sciences

August 24, 2021

Joint work with Assaf Hasson and Kobi Peterzil

# Valued Fields

**Definition**

A valued field $(K, v)$ is a field together with a group homomorphism $v : K^\times \to \Gamma$, where $\Gamma$ is some ordered abelian group, satisfying that

$$v(x + y) \geq \min\{v(x), v(y)\}.$$

(we set $v(0) = \infty$)

## Valued Fields

**Definition**

A valued field $(K, v)$ is a field together with a group homomorphism $v : K^\times \to \Gamma$, where $\Gamma$ is some ordered abelian group, satisfying that

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

(we set $v(0) = \infty$)

**Example**

1. The completion of $\mathbb{Q}$ with respect to the p-adic valuation $v(p^k \frac{m}{n}) = k$. This is the p-adic field $\mathbb{Q}_p$.

# Valued Fields

### Definition

A valued field $(K, v)$ is a field together with a group homomorphism $v : K^\times \to \Gamma$, where $\Gamma$ is some ordered abelian group, satisfying that

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

(we set $v(0) = \infty$)

### Example

1. The completion of $\mathbb{Q}$ with respect to the p-adic valuation $v(p^k \frac{m}{n}) = k$. This is the p-adic field $\mathbb{Q}_p$.
2. $\mathbb{C}((t^{\mathbb{Q}}))$ with $v(\sum a_\gamma t^\gamma) = \min\{\gamma : a_\gamma \neq 0\}$. This is an algebraically closed valued field.

# Valued Fields

## Definition

A valued field $(K, v)$ is a field together with a group homomorphism $v : K^\times \to \Gamma$, where $\Gamma$ is some ordered abelian group, satisfying that

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

(we set $v(0) = \infty$)

## Example

1. The completion of $\mathbb{Q}$ with respect to the p-adic valuation $v(p^k \frac{m}{n}) = k$. This is the p-adic field $\mathbb{Q}_p$.
2. $\mathbb{C}((t^{\mathbb{Q}}))$ with $v(\sum a_\gamma t^\gamma) = \min\{\gamma : a_\gamma \neq 0\}$. This is an algebraically closed valued field.
3. $\mathbb{R}((t^{\mathbb{Q}}))$ is a real closed valued field.

# Valued Fields Cont.

**Definition**

Let $(K, v)$ be a valued field.

# Valued Fields Cont.

**Definition**

Let $(K, v)$ be a valued field.
The valuation ring is the ring $\mathcal{O} = \{x \in K : v(x) \geq 0\}$.

# Valued Fields Cont.

**Definition**

Let $(K, v)$ be a valued field.
The valuation ring is the ring $\mathcal{O} = \{x \in K : v(x) \geq 0\}$. It is a local ring with maximal ideal $\mathcal{M} = \{x \in K : v(x) > 0\}$.

# Valued Fields Cont.

**Definition**

Let $(K, v)$ be a valued field.
The valuation ring is the ring $\mathcal{O} = \{x \in K : v(x) \geq 0\}$. It is a local ring with maximal ideal $\mathcal{M} = \{x \in K : v(x) > 0\}$.
The residue field of $(K, v)$ is the field $k := \mathcal{O}/\mathcal{M}$.

# Valued Fields Cont.

**Definition**

Let $(K, v)$ be a valued field.
The valuation ring is the ring $\mathcal{O} = \{x \in K : v(x) \geq 0\}$. It is a local ring with maximal ideal $\mathcal{M} = \{x \in K : v(x) > 0\}$.
The residue field of $(K, v)$ is the field $k := \mathcal{O}/\mathcal{M}$.
The value group of $(K, v)$ is $\Gamma$.

# History

**Theorem (Poizat)**

*Any infinite field definable in an algebraically closed field $K$ is definably isomorphic to $K$.*

# History

**Theorem (Poizat)**

*Any infinite field definable in an algebraically closed field $K$ is definably isomorphic to $K$. Since algebraically closed fields eliminate imaginaries we also have it for interpretable fields.*

# History

**Theorem (Poizat)**

*Any infinite field definable in an algebraically closed field $K$ is definably isomorphic to $K$. Since algebraically closed fields eliminate imaginaries we also have it for interpretable fields.*

A theorem of a similar nature is true in the o-minimal setting:

**Theorem (Otero-Peterzil-Pillay)**

*Let $\mathcal{R}$ be an o-minimal expansion of a real closed field, $R$. Then any infinite field definable in $\mathcal{R}$ is definably isomorphic to $R$ or to $R(\sqrt{-1})$.*

# History

**Theorem (Poizat)**

*Any infinite field definable in an algebraically closed field $K$ is definably isomorphic to $K$. Since algebraically closed fields eliminate imaginaries we also have it for interpretable fields.*

A theorem of a similar nature is true in the o-minimal setting:

**Theorem (Otero-Peterzil-Pillay)**

*Let $\mathcal{R}$ be an o-minimal expansion of a real closed field, $R$. Then any infinite field definable in $\mathcal{R}$ is definably isomorphic to $R$ or to $R(\sqrt{-1})$. (o-minimal fields eliminate imaginaries)*

# History, cont.

**Theorem (Hrushovski and Rideau-Kikuch)**

*Let $K$ be an algebraically closed valued field. Any interpretable infinite field in $K$ is definably isomorphic to $K$ or to the residue field $k$.*

# History, cont.

**Theorem (Hrushovski and Rideau-Kikuch)**

*Let $K$ be an algebraically closed valued field. Any interpretable infinite field in $K$ is definably isomorphic to $K$ or to the residue field $k$.*

**Theorem (Pillay)**

*Every infinite definable field in $\mathbb{Q}_p$ is definably isomorphic to a finite extension of $\mathbb{Q}_p$.*

# History, cont.

**Theorem (Hrushovski and Rideau-Kikuch)**

*Let $K$ be an algebraically closed valued field. Any interpretable infinite field in $K$ is definably isomorphic to $K$ or to the residue field $k$.*

**Theorem (Pillay)**

*Every infinite definable field in $\mathbb{Q}_p$ is definably isomorphic to a finite extension of $\mathbb{Q}_p$.*

Pillay asked: What about interpretable fields in $\mathbb{Q}_p$?

# History, cont.

**Theorem (Hrushovski and Rideau-Kikuch)**

*Let $K$ be an algebraically closed valued field. Any interpretable infinite field in $K$ is definably isomorphic to $K$ or to the residue field $k$.*

**Theorem (Pillay)**

*Every infinite definable field in $\mathbb{Q}_p$ is definably isomorphic to a finite extension of $\mathbb{Q}_p$.*

Pillay asked: What about interpretable fields in $\mathbb{Q}_p$?

**Theorem**

- *(Sokolovic) An infinite field of finite Morley rank interpretable in $DCF_0$ is definably isomorphic to the field of constants.*
- *(Sklinos) No infinite field is interpretable in the free group.*

# Main Results

We propose a new recipe for providing results of this nature in various dp-minimal (valued) fields.

# Main Results

We propose a new recipe for providing results of this nature in various dp-minimal (valued) fields. For interpretable fields we show:

**Theorem (H.-Hasson-Peterzil)**

*In each of the following cases, every interpretable field is definably isomorphic to a finite extension of either the valued field or the residue field:*

# Main Results

We propose a new recipe for providing results of this nature in various dp-minimal (valued) fields. For interpretable fields we show:

**Theorem (H.-Hasson-Peterzil)**

*In each of the following cases, every interpretable field is definably isomorphic to a finite extension of either the valued field or the residue field:*

1. *P-minimal fields with generic differentiability*

# Main Results

We propose a new recipe for providing results of this nature in various dp-minimal (valued) fields. For interpretable fields we show:

**Theorem (H.-Hasson-Peterzil)**

*In each of the following cases, every interpretable field is definably isomorphic to a finite extension of either the valued field or the residue field:*

1. *P-minimal fields with generic differentiability (p-adically closed fields, e.g. $\mathbb{Q}_q$)*

# Main Results

We propose a new recipe for providing results of this nature in various dp-minimal (valued) fields. For interpretable fields we show:

Theorem (H.-Hasson-Peterzil)

*In each of the following cases, every interpretable field is definably isomorphic to a finite extension of either the valued field or the residue field:*

1. *P-minimal fields with generic differentiability (p-adically closed fields, e.g. $\mathbb{Q}_q$)*
2. *T-convex power bounded valued fields*

# Main Results

We propose a new recipe for providing results of this nature in various dp-minimal (valued) fields. For interpretable fields we show:

**Theorem (H.-Hasson-Peterzil)**

*In each of the following cases, every interpretable field is definably isomorphic to a finite extension of either the valued field or the residue field:*

1. *P-minimal fields with generic differentiability (p-adically closed fields, e.g.* $\mathbb{Q}_q$*)*
2. *T-convex power bounded valued fields (RCVF)*

# Main Results

We propose a new recipe for providing results of this nature in various dp-minimal (valued) fields. For interpretable fields we show:

**Theorem (H.-Hasson-Peterzil)**

*In each of the following cases, every interpretable field is definably isomorphic to a finite extension of either the valued field or the residue field:*

1. *P-minimal fields with generic differentiability (p-adically closed fields, e.g. $\mathbb{Q}_q$)*
2. *T-convex power bounded valued fields (RCVF)*
3. *V-minimal fields*

# Main Results

We propose a new recipe for providing results of this nature in various dp-minimal (valued) fields. For interpretable fields we show:

**Theorem (H.-Hasson-Peterzil)**

*In each of the following cases, every interpretable field is definably isomorphic to a finite extension of either the valued field or the residue field:*

1. *P-minimal fields with generic differentiability (p-adically closed fields, e.g. $\mathbb{Q}_q$)*
2. *T-convex power bounded valued fields (RCVF)*
3. *V-minimal fields ($ACVF_{0,0}$)*

# Main Results - Definable Fields

To illustrate (some) of the techniques we use, we will focus here on definable fields.

# Main Results - Definable Fields

To illustrate (some) of the techniques we use, we will focus here on definable fields. We start with the result:

# Main Results - Definable Fields

To illustrate (some) of the techniques we use, we will focus here on definable fields. We start with the result:

Theorem (H.-Hasson-Peterzil)

*Every infinite definable field in a dp-minimal valued field* $(K, v, \dots)$ *with generic differentiability is definably isomorphic to a finite extension of* $K$.

# Main Results - Definable Fields

To illustrate (some) of the techniques we use, we will focus here on definable fields. We start with the result:

**Theorem (H.-Hasson-Peterzil)**

*Every infinite definable field in a dp-minimal valued field $(K, v, \dots)$ with generic differentiability is definably isomorphic to a finite extension of $K$. In particular, this is true for pure dp-minimal valued fields of characteristic 0.*

# Main Results - Definable Fields

To illustrate (some) of the techniques we use, we will focus here on definable fields. We start with the result:

**Theorem (H.-Hasson-Peterzil)**

*Every infinite definable field in a dp-minimal valued field $(K, v, \dots)$ with generic differentiability is definably isomorphic to a finite extension of $K$.*
*In particular, this is true for pure dp-minimal valued fields of characteristic $0$.*
*As a result, this result is also true for any infinite dp-minimal pure field of characteristic $0$.*

# Main Results - Definable Fields

To illustrate (some) of the techniques we use, we will focus here on definable fields. We start with the result:

**Theorem (H.-Hasson-Peterzil)**

*Every infinite definable field in a dp-minimal valued field $(K, v, \dots)$ with generic differentiability is definably isomorphic to a finite extension of $K$. In particular, this is true for pure dp-minimal valued fields of characteristic 0.*
*As a result, this result is also true for any infinite dp-minimal pure field of characteristic 0.*

The definition of dp-minimality is immaterial at this moment. It is sufficient to know that it implies that these fields have nice topological properties. Generic differentiability holds, e.g., in 1-h-minimal fields.

# Setting: SW-Uniformities

Most of our results are based on a careful analysis of what we call SW-uniformities. For simplicity:

# Setting: SW-Uniformities

Most of our results are based on a careful analysis of what we call SW-uniformities. For simplicity:

- A dp-minimal abelian group $(G, +, 0, \dots)$

# Setting: SW-Uniformities

Most of our results are based on a careful analysis of what we call SW-uniformities. For simplicity:

- A dp-minimal abelian group $(G, +, 0, \dots)$
- together with a uniformally definable basis of neighbourhoods of 0

# Setting: SW-Uniformities

Most of our results are based on a careful analysis of what we call SW-uniformities. For simplicity:

- A dp-minimal abelian group $(G, +, 0, \dots)$
- together with a uniformally definable basis of neighbourhoods of 0
- Hausdorff non-discrete

# Setting: SW-Uniformities

Most of our results are based on a careful analysis of what we call SW-uniformities. For simplicity:

- A dp-minimal abelian group $(G, +, 0, \dots)$
- together with a uniformally definable basis of neighbourhoods of 0
- Hausdorff non-discrete
- no isolated points

# Setting: SW-Uniformities

Most of our results are based on a careful analysis of what we call SW-uniformities. For simplicity:

- A dp-minimal abelian group $(G, +, 0, \ldots)$
- together with a uniformally definable basis of neighbourhoods of 0
- Hausdorff non-discrete
- no isolated points
- every infinite definable subset of $G$ has non-empty interior

# Setting: SW-Uniformities

Most of our results are based on a careful analysis of what we call SW-uniformities. For simplicity:

- A dp-minimal abelian group $(G, +, 0, \dots)$
- together with a uniformally definable basis of neighbourhoods of 0
- Hausdorff non-discrete
- no isolated points
- every infinite definable subset of $G$ has non-empty interior

**Example**

1. weakly o-minimal ordered groups
2. dp-minimal valued fields

## Setting: SW-Uniformities

Most of our results are based on a careful analysis of what we call SW-uniformities. For simplicity:

- A dp-minimal abelian group $(G, +, 0, \dots)$
- together with a uniformally definable basis of neighbourhoods of 0
- Hausdorff non-discrete
- no isolated points
- every infinite definable subset of $G$ has non-empty interior

**Example**

1. weakly o-minimal ordered groups
2. dp-minimal valued fields

The first has nice 1-dimensional definable sets, the second (in the examples) usually satisfies that acl has exchange.

# Setting: SW-Uniformities, cont.

Simon-Walsberg proved that SW-uniformities have nice topological properties:

# Setting: SW-Uniformities, cont.

Simon-Walsberg proved that SW-uniformities have nice topological properties:

1. dp-rank=naive topological dimension=acl-dimension

# Setting: SW-Uniformities, cont.

Simon-Walsberg proved that SW-uniformities have nice topological properties:

1. dp-rank=naive topological dimension=acl-dimension
2. Dimension is definable in families

# Setting: SW-Uniformities, cont.

Simon-Walsberg proved that SW-uniformities have nice topological properties:

1. dp-rank=naive topological dimension=acl-dimension
2. Dimension is definable in families
3. Every definable function is locally continuous

# Setting: SW-Uniformities, cont.

Simon-Walsberg proved that SW-uniformities have nice topological properties:

1. dp-rank=naive topological dimension=acl-dimension
2. Dimension is definable in families
3. Every definable function is locally continuous
4. $\dim(\mathrm{Fr}(X)) < \dim(X)$

# Setting: SW-Uniformities, cont.

Building on the work by Simon-Walsberg we proved further nice topological results (replacing exchange in various arguments):

# Setting: SW-Uniformities, cont.

Building on the work by Simon-Walsberg we proved further nice topological results (replacing exchange in various arguments):

1. For every open $V \subseteq G^n$, $a \in V$ and $A$ a small set of parameters, there exists $B \supseteq A$ and a $B$-definable open subset $U = U_1 \times \cdots \times U_n \subseteq V$ such that $a \in U$ and $\dim(a/B) = \dim(a/A)$.

# Setting: SW-Uniformities, cont.

Building on the work by Simon-Walsberg we proved further nice topological results (replacing exchange in various arguments):

1. For every open $V \subseteq G^n$, $a \in V$ and $A$ a small set of parameters, there exists $B \supseteq A$ and a $B$-definable open subset $U = U_1 \times \cdots \times U_n \subseteq V$ such that $a \in U$ and $\dim(a/B) = \dim(a/A)$.
2. Let $M$ be small model and $b_1, \ldots, b_n$ some tuples. For any $M$-definable $X$, there exists $a \in X$, with $\dim(a/M) = \dim(X)$, such that $\dim(a, b_i/M) = \dim(a/M) + \dim(b_i/M)$ for all $1 \leq i \leq n$.

# Generics

In stable theories one usually uses (the uniqueness of) generic types to study definable fields.

# Generics

In stable theories one usually uses (the uniqueness of) generic types to study definable fields.

In our setting, since the topology is Hausdorff there is no hope for such uniqueness.

# Generics

In stable theories one usually uses (the uniqueness of) generic types to study definable fields.

In our setting, since the topology is Hausdorff there is no hope for such uniqueness.

The solution: Consider (partial) types of infinitesimals.

# Type of Infinitesimals

Let $\mathcal{F} \subseteq K^n$ be a $\emptyset$-definable field in some (sufficiently saturated) SW-uniform field $(K, +, \cdot, \ldots)$.

# Type of Infinitesimals

Let $\mathcal{F} \subseteq K^n$ be a $\emptyset$-definable field in some (sufficiently saturated) SW-uniform field $(K, +, \cdot, \dots)$.
(really – works even if $\mathcal{F}$ is some interpretable field which is only locally embedable into $K^n$ - but then the proofs are a bit more complicated).

# Type of Infinitesimals

Let $\mathcal{F} \subseteq K^n$ be a $\emptyset$-definable field in some (sufficiently saturated) SW-uniform field $(K, +, \cdot, \dots)$.
(really – works even if $\mathcal{F}$ is some interpretable field which is only locally embedable into $K^n$ - but then the proofs are a bit more complicated).

**Definition**

For any $d \in \mathcal{F}$ with $\dim(d/M) = \dim(\mathcal{F})$, where $M$ is a small model, the infinitesimal neighborhood of $d$ is

$$\nu_{\mathcal{F}}(d) = \{U \cap \mathcal{F} : U \subseteq K^n \text{ definable open and } d \in U\}.$$

# Group of Infinitesimals

# Group of Infinitesimals

**Proposition**

1. *For any such d, $\nu_{\mathcal{F}}(d) - d$ is a subgroup of $(\mathcal{F}, +)$. It is independent of d: we name it by $\nu$.*

# Group of Infinitesimals

Proposition

1. *For any such $d$, $\nu_{\mathcal{F}}(d) - d$ is a subgroup of $(\mathcal{F}, +)$. It is independent of $d$: we name it by $\nu$.*
2. *$\nu$ is invariant under multiplication by elements of $\mathcal{F}$.*

# Generic Differentiability

From now on assume that $K$ comes equipped with some definable valuation or norm. This allows us to talk about differentiability of definable functions.

# Generic Differentiability

From now on assume that $K$ comes equipped with some definable valuation or norm. This allows us to talk about differentiability of definable functions.

- We now further assume that $K$ satisfies generic differentiability: that every definable function is locally differentiable.

# Generic Differentiability

From now on assume that $K$ comes equipped with some definable valuation or norm. This allows us to talk about differentiability of definable functions.

- We now further assume that $K$ satisfies generic differentiability: that every definable function is locally differentiable.

The key observations are (for $K \prec \widehat{K}$ sufficiently saturated):

# Generic Differentiability

From now on assume that $K$ comes equipped with some definable valuation or norm. This allows us to talk about differentiability of definable functions.

- We now further assume that $K$ satisfies generic differentiability: that every definable function is locally differentiable.

The key observations are (for $K \prec \widehat{K}$ sufficiently saturated):

1. $\nu(\widehat{K})$ is a $\mathcal{D}^1$-group

# Generic Differentiability

From now on assume that $K$ comes equipped with some definable valuation or norm. This allows us to talk about differentiability of definable functions.

- We now further assume that $K$ satisfies generic differentiability: that every definable function is locally differentiable.

The key observations are (for $K \prec \widehat{K}$ sufficiently saturated):

1. $\nu(\widehat{K})$ is a $\mathcal{D}^1$-group
2. for every $c \in \mathcal{F}$, $\lambda_c : \nu(\widehat{K}) \to \nu(\widehat{K})$ mapping $x \mapsto cx$ is a $\mathcal{D}^1$-homomorphism.

# Wrapping Up

We have a definable function $\mathcal{F} \to M_n(K)$ mapping $c \mapsto J_{\lambda_c}(0)$.

# Wrapping Up

We have a definable function $\mathcal{F} \to M_n(K)$ mapping $c \mapsto J_{\lambda_c}(0)$. By the chain rule it is a homomorphism so injective.

# Wrapping Up

We have a definable function $\mathcal{F} \to M_n(K)$ mapping $c \mapsto J_{\lambda_c}(0)$. By the chain rule it is a homomorphism so injective. We may thus view $\mathcal{F}$ as a subring of $M_n(K)$.

# Wrapping Up

We have a definable function $\mathcal{F} \to M_n(K)$ mapping $c \mapsto J_{\lambda_c}(0)$. By the chain rule it is a homomorphism so injective. We may thus view $\mathcal{F}$ as a subring of $M_n(K)$.
Letting $K_0 = \{aI_n : a \in K\}$, $K_0 \cap \mathcal{F}$ is an infinite definable field (by NIP),

# Wrapping Up

We have a definable function $\mathcal{F} \to M_n(K)$ mapping $c \mapsto J_{\lambda_c}(0)$. By the chain rule it is a homomorphism so injective. We may thus view $\mathcal{F}$ as a subring of $M_n(K)$.
Letting $K_0 = \{aI_n : a \in K\}$, $K_0 \cap \mathcal{F}$ is an infinite definable field (by NIP), but since $K_0$ is dp-minimal necessarily $K_0 \cap \mathcal{F} = K_0$ (so $K_0 \subseteq \mathcal{F}$).

# Wrapping Up

We have a definable function $\mathcal{F} \to M_n(K)$ mapping $c \mapsto J_{\lambda_c}(0)$. By the chain rule it is a homomorphism so injective. We may thus view $\mathcal{F}$ as a subring of $M_n(K)$.
Letting $K_0 = \{aI_n : a \in K\}$, $K_0 \cap \mathcal{F}$ is an infinite definable field (by NIP), but since $K_0$ is dp-minimal necessarily $K_0 \cap \mathcal{F} = K_0$ (so $K_0 \subseteq \mathcal{F}$). Now observe that $\mathcal{F}$ is a finite extension of $K_0$. □

# Recipe for Interpretable Fields

The key observation is that we do not need to understand all the imaginary sorts, i.e.

# Recipe for Interpretable Fields

The key observation is that we do not need to understand all the imaginary sorts, i.e. do not require elimination of imaginaries in some nice language. Really - in many of the examples no such language is known.

# Recipe for Interpretable Fields

The key observation is that we do not need to understand all the imaginary sorts, i.e. do not require elimination of imaginaries in some nice language. Really - in many of the examples no such language is known.

Let $(K, v, \dots)$ be a dp-minimal valued field with generic differentiability and let $\mathcal{F}$ be an interpretable field.

# Recipe for Interpretable Fields, cont.

1. Reduction to Unary Imaginaries:

# Recipe for Interpretable Fields, cont.

1. Reduction to Unary Imaginaries: There exists an infinite definable subset $I \subseteq \mathcal{F}$ that is definably isomorphic to $K/E$, for some definable equivalence relation $E$ on $K$

# Recipe for Interpretable Fields, cont.

1. Reduction to Unary Imaginaries: There exists an infinite definable subset $I \subseteq \mathcal{F}$ that is definably isomorphic to $K/E$, for some definable equivalence relation $E$ on $K$ (this only uses the fact that $\mathcal{F}$ codes finite sets).

# Recipe for Interpretable Fields, cont.

1. Reduction to Unary Imaginaries: There exists an infinite definable subset $I \subseteq \mathcal{F}$ that is definably isomorphic to $K/E$, for some definable equivalence relation $E$ on $K$ (this only uses the fact that $\mathcal{F}$ codes finite sets).
2. Reduction to known sorts:

# Recipe for Interpretable Fields, cont.

1. Reduction to Unary Imaginaries: There exists an infinite definable subset $I \subseteq \mathcal{F}$ that is definably isomorphic to $K/E$, for some definable equivalence relation $E$ on $K$ (this only uses the fact that $\mathcal{F}$ codes finite sets).
2. Reduction to known sorts: In the examples we consider, we find a definable finite-to-one function from (an infinite subset of) $K/E$ to either $\Gamma$, $K/\mathcal{O}$, $k^n$ or $K^n$

# Recipe for Interpretable Fields, cont.

1. Reduction to Unary Imaginaries: There exists an infinite definable subset $I \subseteq \mathcal{F}$ that is definably isomorphic to $K/E$, for some definable equivalence relation $E$ on $K$ (this only uses the fact that $\mathcal{F}$ codes finite sets).
2. Reduction to known sorts: In the examples we consider, we find a definable finite-to-one function from (an infinite subset of) $K/E$ to either $\Gamma$, $K/\mathcal{O}$, $k^n$ or $K^n$ (actually one can assume that this function is injective).

# Recipe for Interpretable Fields, cont.

3. Eliminate unwanted sorts:

# Recipe for Interpretable Fields, cont.

3. Eliminate unwanted sorts:
    - P-minimal: $\Gamma$ and $K/\mathcal{O}$ are “discrete” but $\mathcal{F}$ eliminates $\exists^\infty$.

# Recipe for Interpretable Fields, cont.

3. Eliminate unwanted sorts:
   - P-minimal: $\Gamma$ and $K/\mathcal{O}$ are “discrete” but $\mathcal{F}$ eliminates $\exists^\infty$.
   - T-convex power bounded: $\Gamma$ and $K/\mathcal{O}$ are “linear” (every definable function is locally affine).

# Recipe for Interpretable Fields, cont.

3. Eliminate unwanted sorts:
   - P-minimal: $\Gamma$ and $K/\mathcal{O}$ are “discrete” but $\mathcal{F}$ eliminates $\exists^\infty$.
   - T-convex power bounded: $\Gamma$ and $K/\mathcal{O}$ are “linear” (every definable function is locally affine).
   - V-minimal: $\Gamma$ is a DOAG (so “linear”) and $K/\mathcal{O}$ is “linear”.

# Recipe for Interpretable Fields, cont.

4. $K$ is an SW-uniformity so we may apply an adaptation of the above.

# Recipe for Interpretable Fields, cont.

4. $K$ is an SW-uniformity so we may apply an adaptation of the above.
5. In case of t-convex power bounded, $k$ is also an SW-uniformity.

# Recipe for Interpretable Fields, cont.

4. $K$ is an SW-uniformity so we may apply an adaptation of the above.
5. In case of t-convex power bounded, $k$ is also an SW-uniformity.
6. If $k$ is an algebraically closed field then one needs to proceed by proving a local version of Zil'ber's indecomposability theorem.

# Generalizations

1. Can we use this machinery to analyze interpretable fields in a dp-minimal 1-h-minimal field?

# Generalizations

1. Can we use this machinery to analyze interpretable fields in a dp-minimal 1-h-minimal field?
2. What about in valued fields of higher dp-rank?

# Thank You!