Pillay's conjecture for groups definable in weakly o-minimal nonvaluational structures

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  Qn: Extensions?

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In (a): is $\mathcal{M}$ the trace of a real closed field $\overline{\mathcal{M}}$?

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There is o-minimal $\overline{\mathcal{M}} = \langle \overline{M}, <, +, \ldots \rangle$, such that $X \subseteq M^n$ is definable $\Leftrightarrow$ $X = Y \cap M^n$, for some $\overline{\mathcal{M}}$-definable set $Y \subseteq \overline{M}^n$.

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Bar-Yehuda, Hasson, Peterzil: $\overline{\mathcal{M}} = \big\langle \overline{M}, \{cl(X)\}_{X \subseteq M^n \text{ is } \emptyset\text{-definable}} \big\rangle$.

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3. *Every definable map extends to an $\overline{\mathcal{M}}$-definable map.*

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Definition

Let $\mathcal{N}$ be o-minimal, and $V \subseteq X \subseteq N^n$ definable. We call V *large in X* if $\dim(X \setminus V) < \dim X$.

Group chunk ($\mathcal{N}$ o-minimal)

- An $\mathcal{N}$*-definable group chunk* is a triple triple (X, i, F), where $X \subseteq N^n$ is $\mathcal{N}$-definable, and $i : X \to N^n$, $F : Z \subseteq X^2 \to N^n$ are $\mathcal{N}$-definable maps that "resemble" group inverse and multiplication, respectively, on a large subset of X:

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3. for every $(x, y) \in Z$, there is a large $S_{(x,y)} \subseteq X$, such that for all $z \in S_{(x,y)}$, the following expressions are defined and are equal:

$$(a)\quad F(F(x, y), z) = F(x, F(y, z))$$
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4. for every $x \in X$, $\pi_1 F^{-1}(x)$ and $\pi_2 F^{-1}(x)$ are large in X.

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Let (X, i, F) be an $\mathcal{N}$-definable group chunk, with $X \subseteq N^n$, $i : X \to N^n$, and $F : Z \subseteq X^2 \to N^n$.

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(i) *there is an $\mathcal{N}$-definable group $K = \langle K, * \rangle$ with $X \subseteq K$ large in K, and such that for every $(x, y) \in Z$,*

$$F(x, y) = x * y.$$

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(ii) *K $\mathcal{N}$-definably embeds into any $\mathcal{N}$-definable group H that extends (X, i, F), and if X is large in H, then $K \cong H$ (in $\mathcal{N}$).*

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Sketch of the proof.

Step I. Find an $\overline{\mathcal{M}}$-definable group chunk (X, i, F), extending $(G, \cdot)$, with $G \subseteq X \subseteq cl(G)$.

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Pf. Extend $^{-1}, \cdot$ to $\overline{\mathcal{M}}$-definable maps $i : X \to N^n$, $F : Z \subseteq X^2 \to \overline{M}^n$, making sure i, $F(x, -)$ and $F(-, x)$ remain injective.

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Also, $\dim G = \dim cl(G) = \dim X = \dim K$.

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fsg – measures

Fact

Let G be a group definable in a NIP structure. TFAE:

1. *G has fsg*
2. *G admits a left-invariant generically stable measure*
3. *G admits a left-invariant finitely interpretation measure (fim).*

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For 3, if μ is a nice measure on G, so is $\nu(X) := \mu(X \cap G)$ on K. □

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Thank you!

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A *Keisler measure* μ on G is a finitely additive probability measure on $\text{Def}(G)$; that is, a map $\mu : \text{Def}(G) \to [0, 1]$ such that $\mu(\emptyset) = 0$, $\mu(G) = 1$, and for $Y, Z \in \text{Def}(G)$,

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$$\mu(Y \cup Z) = \mu(Y) + \mu(Z) - \mu(Y \cap Z).$$

For $X \subseteq M^r$, $a_1, \ldots, a_k \in M^r$, $\mathrm{Av}(a_1, \ldots, a_k; X)| := \frac{1}{k}|\{i : a_i \in X\}|$.

A Keisler measure μ on G is called a *frequency interpretation measure (fim)* if for every formula $\varphi(x; y)$ and $\varepsilon > 0$, there are $a_1, \ldots, a_k \in M^n$, such that for every $c \in M^m$, and for $X = \varphi(R; c)$, we have

$$|\mu(X) - \mathrm{Av}(a_1, \ldots, a_k; X)| < \varepsilon.$$

Example

Let $G = S^1$, μ measures length, and $\mu(G) = 1$.
If $\varphi(x, a, b)$ defines the arc (a, b), and $\varepsilon \leq 1$,
we can let $k = \ulcorner 1/\varepsilon \urcorner$ and choose $a_1, \ldots, a_k$ to equipartition G.