Decompositions of groups definable in o-minimal structures

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Some references

- L. van den Dries, **Tame topology and o-minimal structures**, LMS Lecture Note Series 248, Cambridge University Press, 1998.

- D. Macpherson, **Notes on o-minimality and variations**, in Model Theory, Algebra and Geometry, ed. Haskell, Pillay and Steinhorn, MSRI publications, Cambridge University Press, 2000, pp. 97–130. pdf

- Y. Peterzil, **A self-guide to o-minimality**, 2007. pdf

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- M. Otero, **A survey on groups definable in o-minimal structures**, in Model Theory with Applications to Algebra and Analysis, vol. 2, Editors: Z. Chatzidakis, D. Macpherson, A.Pillay and A. Wilkie, LMS LNS 350, Cambridge University Press, 2008, pp. 177–202. pdf

- A. Conversano, **Groups definable in o-minimal structures: various properties and a diagram**, 2021. pdf

o-minimal structures

Definition

$\mathcal{M} = (M, <, \dots)$ is o-minimal when every definable set $X \subset M$ is a finite union of points and open intervals.

Remark

M has the order topology,
M^k has the product topology

Examples

- $(\mathbb{Q}, <, +)$ and any other ordered divisible abelian gr.
- $(\mathbb{R}, <, +, \cdot)$ and any other real closed field (i.e. ordered field M with $M(\sqrt{-1})$ alg. closed)
- $\mathbb{R}_{exp} = (\mathbb{R}, <, +, \cdot, \exp)$ (Wilkie)
- $\mathbb{R}_{an,exp}$ (van den Dries - Miller)

Definition

Let X be a definable set and let $\mathcal{P}$ be a partition of X in cells C. We denote by $E(X)$ the **o-minimal Euler characteristic** of X, which is given by

$$E(X) = \sum_{C \in \mathcal{P}} E(C) = \sum_{C \in \mathcal{P}} (-1)^{\dim C}$$

Theorem

Let X, Y be definable set in an o-minimal expansion of a r.c.f. Then there is a definable bijection between X and Y if and only if

$$\dim X = \dim Y \quad \textit{and} \quad E(X) = E(Y)$$

Definable groups and induced topology from the structure

$(G, \cdot)$ is a definable group in $\mathcal{M}$ when G and $\Gamma(\cdot)$ are definable sets in $\mathcal{M}$.

Example

Let $\mathcal{M} = (M, <, +, \dots)$ be an o-minimal expansion of an ordered group. Fix $a > 0$ and take $G = [0, a)$ with the group operation

$$x * y = \begin{cases} x + y & \text{if } x + y < a \\ x + y - a & \text{otherwise.} \end{cases}$$

The group operation is not continuous in 0 with respect to the topology induced by $<$

To make it continuous, set $\{U_\varepsilon = [0, \varepsilon) \bigcup (a - \varepsilon, a) : \varepsilon > 0\}$ a basis for the neighborhoods of 0.

Moreover, if $\mathcal{M} = (M, <, +, \cdot, \dots)$ expands a field, then we can definably embed $(G, *)$ into M^2 (take $a = 1$) with the map:

$$x \mapsto \begin{cases} (-4x + 1, \sqrt{1 - (4x - 1)^2}) & \text{if } x \in [0, 1/2) \\ (4x - 3, \sqrt{1 - (4x - 3)^2}) & \text{if } x \in [1/2, 1). \end{cases}$$

The group image, endowed with the topology induced by M^2, is homeomorphic to $(G, *)$.

Pillay's manifold on definable groups

Theorem (Pillay, 1988)

Let $(G, \cdot)$ be a group definable in an o-minimal structure $\mathcal{M} = (M, <, \dots)$. Then

- *There is a topology τ on G such that $(G, \cdot, \tau)$ is a top group.*
- *There are τ-open definable subsets $U_1, \dots, U_r$ covering G, each definably homeomorphic to an open subset of $M^{\dim G}$*

Fact

Every group definable in an o-minimal expansion of a field is affine. (That is, we can assume τ is the topology induced by the structure)

Definition

A real Lie group of dimension n is a smooth n-dimensional manifold over $\mathbb{R}$, equipped with a smooth group operation.

Corollary (Hilbert 5th)

Every group definable in an o-minimal structure over $\mathbb{R}$ is a real Lie group.

Examples (of real Lie groups)

- Any closed $G < GL_n(\mathbb{R})$ and (assuming $G = G^0$) its universal cover $\widetilde{G}$.

Question

- Which real Lie groups are definable in an o-minimal expansion of the real field?

Classifying real Lie groups

Theorem (C., Onshuus, Post - 2021)

Let G be a real Lie group. Then the following are equivalent:

- *G is Lie isomorphic to a group definable in an o-minimal expansion of the reals.*
- *G is Lie isomorphic to a group definable in $\mathbb{R}_{exp}$.*
- *G and $Z(G)$ have finitely many connected components and its solvable radical is Lie isomorphic to $N \rtimes SO_2(\mathbb{R})^d$, where N is simply connected and completely solvable.*

Theorem (C., Mamino - 2021)

There is a connected nilpotent Lie group that inteprets $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$

Theorem (C. - 2021)

A nilpotent real Lie group G has an o-minimal copy $\iff$ G is isomorphic to a linear algebraic group.

Questions

- Are there other nilpotent groups in between or is it a dichotomy?
  For instance, is there a nilpotent group that is definable in a NIP structure and is not isomorphic to an algebraic group?

- For the groups with an o-minimal copy: which ones are semialgebraic and which ones need the exponential function?

- Are $(\mathbb{R}, +, \cdot)$, $\mathbb{R}_{\text{exp}}$, $\langle \mathbb{R}_{\text{exp}}, (\mathbb{Z}, +) \rangle$, $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$ the only relevant structures?
  For instance, is there a NIP structure that defines more than o-minimal groups (up to Lie isomorphism) and central extensions of semisimple groups?

Topological notions

In a $\aleph_0$-saturated ordered structure, $[a, b]$ is not connected nor compact.

We need different notions that capture connectedness and compactness.

Definition

A definable set X is definably connected if there are no open definable subsets A, B such that $X \subset A \cup B$ and $A \cap B = \emptyset$.

In a definable group G the definably connected component of e is denoted by G^0.

Theorem (Pillay, 1988)

Let G be a definable group. Then G^0 is the smallest (normal) definable subgroup of G of finite index. Thus

$$G \text{ is definably connected} \iff G = G^0.$$

(Peterzil-Steinhorn, 1999)

A definable set X is definably compact if X (with the topology induced by the ambient space) is closed and bounded.

Semisimple groups and the solvable radical

Definition

Let G be a non-abelian definable group.

- G is definably simple if the only normal definable subgr. of G are $\{e\}$ and G.
- G is semisimple if G does not have infinite abelian ($\Leftrightarrow$ solvable) - definable or not - normal subgroups.

Theorem (Peterzil, Pillay, Starchenko - 2000)

Let G be a semisimple definably connected group in an o-minimal structure $\mathcal{M}$. Then $G/Z(G)$ is a direct product of definably simple groups H_i, and each H_i is definably isomorphic to a definable subgroup of $\mathrm{GL}_n(\mathcal{R}_i)$, for some real closed field $\mathcal{R}_i$ definable in $\mathcal{M}$.

Fact

Every definable group G has a maximal normal definably connected solvable subgroup R (called the solvable radical) and the quotient G/R is a semisimple definable group.

Question

- Is there a definable semisimple S such that $G = RS$?

Levi decomposition

Levi decomposition of Lie groups

Let L be a connected Lie group. If R is the solvable radical of L, then there is a connected semisimple subgroup S of L such that

$$L = R \cdot S \qquad \text{and} \qquad \dim(R \cap S) = 0.$$

If L is compact, then its Levi decomposition is $L = Z(L)^0 \cdot [L, L]$

Theorem (Peterzil, Pillay, Starchenko - 2002)

If G is linear, then G has a definable Levi decomposition.

Theorem (Hrushovski, Peterzil, Pillay - 2011)

If G is definably compact, then $[G, G]$ is def. and $G = Z(G)^0 \cdot [G, G]$.

Theorem (C., Pillay - 2013)

G has a maximal ind-definable semisimple subgroup S, unique up to conjugation, and

$$G = R \cdot S$$

- *R being the solvable radical of G,*
- *$R \cap S \subseteq Z(S)$, and $Z(S)$ is finitely generated.*

Definition

S is **ind-definable semisimple** if

- S is ind-definable ($\bigvee$-def, locally def) definably connected
- $S/Z(S)$ is definable semisimple

Iwasawa decomposition of semisimple groups, $G = KAN$

Example

Every matrix with determinant 1 can be decomposed as a product of an orthogonal matrix, a diagonal matrix and a unipotent matrix. For example, for 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \lambda & \lambda x \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Theorem (C. - 2014)

Let G be a definably simple group in an o-minimal structure. Then there exists a definable real closed field $\mathcal{R}$ and some $m \in \mathbb{N}$ such that G is definably isomorphic to a definable group $G_1 < GL_m(\mathcal{R})$, with the following properties:

- (i) $G_1 = KH$, *with* $K = G_1 \cap O_m(\mathcal{R})$ *and* $H = G_1 \cap T_m^+(\mathcal{R})$,
- (ii) $H = AN$, *with* $A = G_1 \cap D_m^+(\mathcal{R})$ *and* $N = G_1 \cap UT_m(\mathcal{R})$,
- (iii) G_1 *(and therefore G) has maximal definably compact subgroups, all definably connected and conjugate to* K.

Corollary

Let G be a connected semisimple group definable in an o-minimal structure. Then G has maximal definably compact subgroups and they are all conjugate to each other. For each such maximal compact subgroup K there is a (maximal) torsion-free definable subgroup H such that $G = KH$ and $K \cap H = \{e\}$. Moreover, $H = AN$ where A is abelian, N is nilpotent and $A \cap N = \{e\}$.

Jordan-Chevalley decomposition

Jordan-Chevalley decomposition of linear algebraic groups

Let G be a connected linear algebraic group. Then

$$G = N \rtimes K$$

where N is a closed contractible torsion-free group (the unipotent radical of G) and K is a central extension of a semisimple group $S = [K, K]$ by an algebraic torus $T = Z(K)^0$.

Theorem (C. - 2021)

Let G be a definably connected group in an o-minimal expansion of a real closed field and S a Levi subgroup of G. Then G can be decomposed as

$$G = NTS$$

where:

- *$N = \mathcal{N}(G)$ is the maximal normal definable torsion-free subgroup of G, T is a maximal abstract torus of the solvable radical centrealizing S, NS is a normal subgroup of G containing all Levi subgroups of G, $N \cap T = \{e\}$ and $N \cap S$ is a central finitely generated group.*
- *S is definable $\iff$ $N \cap S = \{e\}$ and $T \cap S$ is finite.*
- *N has a complement in G (definable or not) $\iff$ TS is definable.*

Sketch of the proof

Strzebonski - 1994

Let G be a definable group. G is a *p-group* if:

- p is a prime number and for any proper definable $H < G$,

$$E(G/H) \equiv 0 \mod p$$

- $p = 0$ and for any proper definable subgroup $H < G$,

$$E(G/H) = 0$$

A maximal p-subgroup of a definable group G is called *p-Sylow*.

Each p-subgroup is contained in a p-Sylow, and p-Sylows are all conjugate.

Take a Levi decomposition $G = RS$ and assume S is definable.

- Step 1: Every 0-Sylow A of $N_G(S)$ is a 0-Sylow of G.
- Step 2: There is an abstract torus $T < A$ such that $N \rtimes T = R$.
- Step 3: T centralizes S.
- Step 4: Generalize to S not definable using $G/Z(G)$ has definable Levi subgroups.

Nilpotency: classical groups vs definable groups

Theorem

Let G be a finite group. Then TFAE:

(a) *G is nilpotent.*

(b) *G has one p-Sylow subgroup for each p dividing $|G|$.*

(c) *All Sylow subgroups of G are normal.*

(d) *G is the direct product of its Sylow subgroups.*

(e) *G has no proper self-normalizing subgroup.*

Theorem

Let G be a linear nilpotent connected Lie group. Then

$$G = N \times T$$

where N is simply-connected torsion-free and T is the maximal torus.

Theorem (C - 2021)

Let G be a definable group such that $\mathcal{N}(G)$ is nilpotent.

(1) *Assume $E(G) \neq 0$. Then TFAE:*

- (a) *G is nilpotent.*
- (b) *G has exactly one p-Sylow subgroup for each prime p dividing $E(G)$.*
- (c) *All p-Sylow subgroups of G are normal.*
- (d) *$G = \mathcal{N}(G) \times H$, where H is the direct product of its (unique) p-Sylow subgroups.*

(2) *Suppose $E(G) = 0$ and $G = G^0$. Then TFAE:*

- (a) *G is nilpotent.*
- (b) *G has exactly one 0-Sylow subgroup.*
- (c) *All 0-Sylow subgroups of G are normal.*
- (d) *$G = \mathcal{N}(G) \times T$, for each T maximal abstract torus of G.*

(3) *Let G be definably connected. Then TFAE:*

- (a) *G is nilpotent.*
- (e) *Every proper definable $H < G$ is contained properly in its normalizer.*

Some questions

definable = definable with parameters in an o-minimal structure

(1) Is there a torsion-free non-nilpotent definable group with no proper self-normalizing (definable) subgroup?

(2) Is every nilpotent torsion-free definable group elementarily equivalent to a linear algebraic group (with same dimension)?

(3) Which definable groups are elementarily equivalent to a linear algebraic group (with same dimension)?

Some references and links

- E. Baro, A. Berarducci and M. Otero, **Cartan subgroups and regular points of o-minimal groups**, Journal of the London Mathematical Society, 100 (2019), 361–382 pdf
- A. Conversano, **A reduction to the compact case for groups definable in o-minimal structures**, Journal of Symbolic Logic, 79 (2014), 45–53. pdf (Iwasawa decomposition)
- A. Conversano, **Nilpotent groups, o-minimal Euler characteristic, and linear algebraic groups**, Journal of Algebra, doi: 10.1016/j.jalgebra.2021.08.004 pdf
- A. Conversano and M. Mamino, **One Lie group to define them all**, to appear in Notre Dame Journal of Formal Logic pdf
- A. Conversano, A. Onshuus and S. Post, **Real Lie groups and o-minimality**, pdf.
- A. Conversano and A. Pillay, **Connected components of definable groups and o-minimality I**, Advances in Mathematics, 231 (2012), 605–623. pdf
- A. Conversano and A. Pillay, **On Levi subgroups and the Levi decomposition for groups definable in o-minimal structures**, Fundamenta Mathematicae, 222 (2013), 49-62. pdf
- E. Hrushovski, Y. Peterzil and A. Pillay, **On central extensions and definably compact groups in o-minimal structures**, J. Algebra, 327 (2011), 71–106.