Forbidden Induced Subgraphs and the Łoś-Tarski Theorem

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A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$.

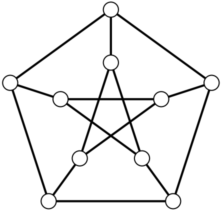


Figure: The Petersen graph

The degree of a vertex $u \in V(G)$ is $\deg(u) := \left|\{v \mid uv \in E(G)\}\right|$. Thereby, the degree of G is

$$\deg(G) := \max_{u \in V(G)} \deg(u).$$

Let $d \geq 1$. A graph G has degree bounded by d if

$$\deg(G) \leqslant d.$$

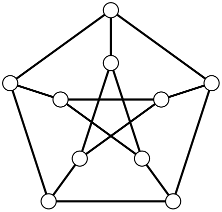


Figure: The Petersen graph has degree bounded by 3.

If G has degree bounded by d and H is an induced subgraph of G, i.e.,

$$V(H) \subseteq V(G) \quad \text{and} \quad E(H) = E(G) \restriction V(H),$$

then H has degree bounded by d as well.

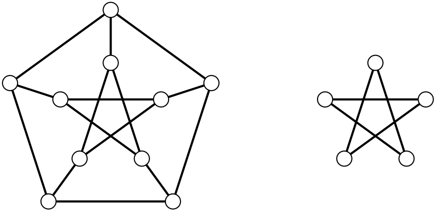
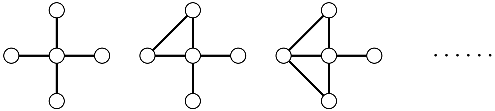


Figure: An induced subgraph of the Petersen graph.

G has degree bounded by 3 if and only if G has no induced subgraph isomorphic to



Graph-theorists are interested in such characterizations, akin to forbidden minors.

Planar graphs have a forbidden minor characterization, but no forbidden induced subgraph characterization.

A graph G has degree bounded by 3 if and only if

$$G \models \forall x \forall y_1 \forall y_2 \forall y_3 \forall y_4 \Big(\bigwedge_{i \in [4]} Exy_i \to \bigvee_{1 \leqslant i < j \leqslant 4} y_i = y_j \Big).$$

This is a universal sentence.

It has been realized rather recently [Sankaran, 2019; *C.* and Flum, 2020] that there is an intimate connection between

- the Łoś-Tarski theorem from classical model theory and
- forbidden induced subgraph characterizations of some natural graph classes.

The motivation (2)

Does logic help us to understand forbidden induced subgraphs?

- We used the logic machinery to provide forbidden induced subgraph characterizations of the classes of graphs with small vertex cover, of bounded tree-depth, and of bounded shrub-depth [*C.* and Flum, 2020].
- However, those characterizations are computed using the Completeness theorem, thus without any explicit bound. Can we do better?

Is there any limitation of such connection?

- It is known that the Łoś-Tarski theorem fails on finite structures [Tait, 1959], but the graph-theoretic version was open.

The plan

1. The connection between forbidden induced subgraph characterization and the Łoś-Tarski theorem.
2. The failure of the Łoś-Tarski theorem, from finite structures [Tait's theorem] to finite graphs [our result].
3. Our proof technique and its further applications on the limitations of the logic methods.

Let $d \geq 1$ and $\mathscr{C}_d := \{ G \mid \text{the degree of } G \text{ is bounded by } d \}$. Then $\mathscr{C}_d$ has two equivalent characterizations.

Logic. For any graph G

$$G \in \mathscr{C}_d \iff G \models \underbrace{\forall x \forall y_1 \ldots \forall y_{d+1} \Big(\bigwedge_{i \in [d+1]} Exy_i \to \bigvee_{1 \leqslant i < j \leqslant d+1} y_i = y_j \Big)}_{\text{universal}}.$$

Forbidden induced subgraphs. Define

$$\mathscr{F}_d := \{ G \mid V(G) = [d+2] \text{ and } G \text{ has a vertex of degree } d+1 \}$$

For any graph G

$$G \in \mathscr{C}_d \iff G \text{ has no induced subgraph isomorphic to a graph in } \mathscr{F}_d.$$

Universal sentences and forbidden induced subgraphs

A first-order logic (FO) sentence φ is universal if it has the form

$$\varphi = \forall x_1 \dots \forall x_k \psi,$$

where ψ is quantifier-free. Define

$$\mathscr{F}_\varphi := \{ G \mid V(G) \subseteq [k] \text{ and } G \not\models \varphi \}.$$

For any graph G

$$G \models \varphi \iff G \text{ has no induced subgraph isomorphic to a graph in } \mathscr{F}_\varphi.$$

The converse is also easy.

A forbidden induced subgraph characterization is equivalent to the definability by a universal sentence

However, there are non-universal sentences that can define classes of graphs with a forbidden induced subgraph characterization.

Let $k \geq 1$ and $\mathscr{C}_k := \{G \mid G \text{ has a vertex cover of size at most } k\}$. Then $\mathscr{C}_k$ has two equivalent characterizations.

Logic. For any graph G

$$G \in \mathscr{C}_k \iff G \models \underbrace{\exists x_1 \ldots \exists x_k \forall u \forall v \Big(Euv \to \bigvee_{i \in [k]} (u = x_i \vee v = x_i) \Big)}_{\text{non-universal}}.$$

Forbidden induced subgraphs. [attributed to Lovász] There is a finite class $\mathscr{F}_k$ of graphs such that for any graph G

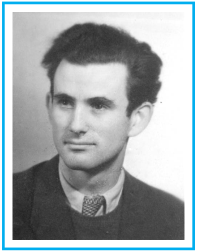
$$G \in \mathscr{C}_k \iff G \text{ has no induced subgraph isomorphic to a graph in } \mathscr{F}_k.$$

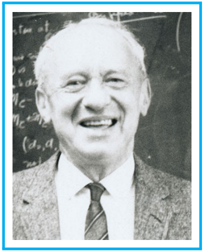
Corollary

For every k there is a universal sentence φ_k such that for any graph G

$$G \models \varphi_k \iff G \textit{ has a vertex cover of size at most } k.$$

The Łoś-Tarski theorem (the graph-theoretic version)





Theorem

Let $\mathscr{C}$ be a class of graphs that is closed under induced subgraphs. Then the following are all equivalent.

1. *$\mathscr{C}$ is definable in* FO.
2. *$\mathscr{C}$ can be defined by a universal sentence.*
3. *$\mathscr{C}$ has a finite set of forbidden induced subgraphs.*

I've cheated . . .

When we think of graphs of bounded degree, with a small vertex cover, . . . , graphs are usually finite.

In the Łoś-Tarski Theorem, graphs can be finite and infinite.

The notions of bounded degree, vertex cover, . . . , can be easily generalized to infinite graphs.

Theorem (Tait, 1959)

There is an FO*-sentence φ which is preserved under induced substructures in finite, i.e., for any finite structures $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A}$ is an induced substructure of $\mathcal{B}$*

$$\mathcal{B} \models \varphi \implies \mathcal{A} \models \varphi,$$

such that φ is not equivalent to any universal sentence.

Remark

Tait's examples might be viewed as ordered colored directed graphs.

Theorem

There is a class $\mathscr{C}$ of finite graphs satisfying the following properties.

1. *$\mathscr{C}$ is closed under induced subgraphs.*
2. *There is an FO-sentence φ such that for every finite graph G*

$$G \in \mathscr{C} \quad \Longleftrightarrow \quad G \models \varphi.$$

3. *$\mathscr{C}$ cannot be characterized by a finite set of forbidden induced subgraphs. Equivalently, φ is not equivalent to any universal sentence.*

1. When applying the Łoś-Tarski theorem to obtain forbidden induced subgraph characterizations of graphs of small vertex cover, bounded tree-depth, bounded shrub-depth [*C.* and Flum, 2020] etc., we really need to extend those notions to infinite graphs.

2. The techniques we've developed enable us to prove a number of results to explain why finding forbidden induced subgraphs might be a hard problem.

   The explicit construction of the forbidden induced subgraphs is only known for tree-depth at most 3 [Dvorak, Giannopoulou, and Thilikos, 2012].

Built on [Gurevich, 1984]

Theorem

For any computable function $f : \mathbb{N} \to \mathbb{N}$, e.g., $f(x) = 2^{2^x}$, there is a class $\mathscr{C}$ of graphs which is closed under induced subgraphs and definable by an FO*-sentence φ such that:*

For any forbidden induced subgraph characterization of $\mathscr{C}$ by

$$\{H_1, \ldots, H_m\}.$$

we have

$$\max_{i \in [m]} |H_i| \geq f(|\varphi|).$$

K has a very succinct description by FO, but gigantic (minimal) forbidden induced subgraphs.

For every φ we define

$$\mathsf{Mod}_{\mathsf{fin}}(\varphi) = \{G \mid \text{finite graph } G \models \varphi\}.$$

Theorem

There is no algorithm that for any φ whose $\mathsf{Mod}_{\mathsf{fin}}(\varphi)$ can be characterized by a finite set of forbidden induced subgraphs computes such a set of forbidden induced subgraphs.

Compared to:

Theorem (using the Completeness theorem)

There is an algorithm that for any φ whose

$$\mathsf{Mod}(\varphi) = \{G \mid \textit{graph } G \models \varphi\}.$$

can be characterized by a finite set of forbidden induced subgraphs computes such a set of forbidden induced subgraphs.

Our techniques

1. We transfer Tait's Theorem and Gurevich's Theorem on arbitrary structures to graphs. In logic, this is done by FO-interpretations.
2. An FO-interpretation I translates any graph G to a structure $\mathcal{A} = \mathcal{A}_I(G)$ such that for every FO-sentence φ there is an FO-sentence φ^I with

   $$\mathcal{A} \models \varphi \iff G \models \varphi^I.$$

   Any property of $\mathcal{A}$ can be captured by a property in G.
3. We need that if G is an induced subgraph of H then $\mathcal{A}_I(G)$ is a induced substructure of $\mathcal{A}_I(H)$. This is not true for the existing FO-interpretations.
4. We introduce the notion of strongly existential interpretations, which preserves the closure of induced substructures/graphs. It requires some technical work and graph gadgets to design desired strongly existential interpretations.

Strongly existential interpretation

An interpretation *I* is strongly existential if all formulas of *I* are existential and in addition $\varphi_<$, i.e., the formula defining a total order, is quantifier-free.

(i) Forbidden induced subgraphs characterization of a class $\mathscr{C}$ of graphs is equivalent to
  - $\mathscr{C}$ is closed under induced subgraphs,
  - and $\mathscr{C}$ is definable by an FO-sentence φ (φ is not necessarily universal).

(ii) One can compute a set of forbidden induced subgraphs $H_1, \ldots, H_m$ for $\mathscr{C}$ from φ.

(iii) An important caveat is that (i) only holds when graphs can be finite or infinite. Otherwise, we've exhibited a class of finite graphs which is closed under induced subgraphs and definable in FO, but has no finite set of forbidden induced subgraphs.

(iv) For (ii) we've proved that $H_1, \ldots, H_m$ can be arbitrarily complex compared to φ. Moreover, if we only consider finite graphs, $H_1, \ldots, H_m$ cannot even be computed from φ.

Thank You!