Structures of Aronszajn/coherent trees

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November 6, 2020

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Theorem 1 (Baumgartner)

Assume PFA. Any two ω_1-dense subsets of reals are isomorphic.

Theorem 2 (Abraham-Shelah)

Assume PFA. Any two normal Aronszajn trees are club isomorphic.

Theorem 3 (Moore)

Assume PFA. There is a five element basis for uncountable linear orders.

Basis problems

Fact 1

CH implies that the basis for uncountable subset of reals has size 2^{ω_1}*.*

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Is it consistent that the least basis size of uncountable linear orders (Aronszajn trees) is different from 5 (1) or 2^{ω_1}?

(Abraham-Shelah) It is consistent that the basis for Suslin trees has size ω_1.

Question 2

Is it consistent to have a small basis for Suslin trees? of size 1?

Basis problems

Theorem 4

Assume the consistency of a supercompact cardinal. Then for any positive integer n, it is consistent that the basis for Aronszajn trees has size n and the basis for uncountable linear orders has size $2n + 3$.

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Definition 1

An Aronszajn tree $T \subset \mathbb{Q}^{<\omega_1}$ is coherent if for any $s, t \in T$, $\{\xi < ht(s), ht(t) : s(\xi) \neq t(\xi)\}$ is finite.

$(T, <_{lex})$ is a linear order where $<_{lex}$ is the lexicographical order.

$(T, <_{lex})$ and $-(T, <_{lex})$ have no uncountable common suborder.

Tree isomorphism

A bijection $\pi : (S, <_S) \to (T, <_T)$ is a *tree isomorphism* if for any $s, t \in S$, $s <_S t$ iff $\pi(s) <_T \pi(t)$.

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Equivalently, for any $s, t \in S$, $\Delta(s, t) = \Delta(\pi(s), \pi(t))$ where $\Delta(s, t) = \max\{\alpha \leq ht(s), ht(t) : s \restriction_\alpha = t \restriction_\alpha\}$.

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Lemma 1

Suppose that T is coherent and $\pi : T \to T$ is a level preserving map, i.e., $ht(\pi(s)) = ht(s)$. Then π induces a tree isomorphism on a subtree.

Tree isomorphism

Proof.

Fix, for each limit α, $s_\alpha \in T_\alpha$. $D_{s_\alpha,\pi(s_\alpha)} = \{\xi < \alpha : s_\alpha(\xi) \neq \pi(s_\alpha)(\xi)\}$ is finite and hence bounded below α.
By the Pressing Down Lemma, we can find $\alpha_0 < \omega_1$ and a stationary subset Γ such that for any $\gamma \in \Gamma$, $D_{s_\gamma,\pi(s_\gamma)} \subset \alpha_0$. So for any $\gamma \in \Gamma$,

$$s_\gamma \restriction_{[\alpha_0,\gamma)} = \pi(s_\gamma) \restriction_{[\alpha_0,\gamma)} . \tag{1}$$

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Going to a stationary subset $\Gamma' \subset \Gamma$, we can find $s, t \in T_{\alpha_0}$ such that for any $\gamma \in \Gamma'$,

$$s_\gamma \restriction_{\alpha_0} = s \text{ and } \pi(s_\gamma) \restriction_{\alpha_0} = t. \tag{2}$$

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Now by (1) and (2), for any $\alpha < \beta$ in Γ', $\Delta(s_\alpha, s_\beta) = \Delta(\pi(s_\alpha), \pi(s_\beta))$. So π induces a tree isomorphism from $T_{\{s_\alpha:\alpha\in\Gamma'\}}$ to $T_{\{\pi(s_\alpha):\alpha\in\Gamma'\}}$. □

Club isomorphism

Example 1

T is coherent. $T^{(1)}$ is the downward closure of $\{s^{(1)} : s \in T\}$ where

$$s^{(1)}(\alpha) = \begin{cases} 0 & \textit{if } \alpha \textit{ is limit} \\ s(\alpha - 1) & \textit{if } \alpha \textit{ is successor and } \alpha \leq ht(s) \end{cases}$$

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Then T is not tree isomorphic to $T^{(1)}$.

$T \restriction_C = \{s \in T : ht(s) \in C\}$ is T's club-restriction subtree where C is a club.

Two trees S, T are *club isomorphic* if they are isomorphic when restricted to a club.

Relations on coherent trees

Definition 2

Suppose S, T are coherent trees.

1. $S \equiv_C T$ *if* $S_X \restriction_D$ *is isomorphic to* $T_Y \restriction_D$ *for some* $X \in [S]^{\omega_1}$, $Y \in [T]^{\omega_1}$ *and club* D.
2. $S <_C T$ *if for any club* D*, there is an uncountable partial level preserving map* $\pi : S \to T$ *such that for any incomparable* s, t *in* $dom(\pi)$, $\Delta_D(s,t) < \Delta_D(\pi(s), \pi(t))$.
3. $S \leq_C T$ *if for some club* D*, there is an uncountable partial level preserving map* $\pi : S \to T$ *such that for any* s, t *in* $dom(\pi)$, $\Delta_D(s,t) \leq \Delta_D(\pi(s), \pi(t))$.

$$\Delta_D(s,t) = \max(D \cap (\Delta(s,t)+1)).$$

Fact 3

Assume MA_{ω_1}. *Every coherent tree can be embedded into any of its subtree.*

Relations on coherent trees

Fact 4

1. *$\equiv_C$ is an equivalence relation on the class of coherent trees.*
2. *$\leq_C$ and $<_C$ are partial orders on equivalence classes $\equiv_C$.*
3. *If $S \leq_C T$, $T \leq_C S$, then $S \equiv_C T$.*

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3. *If* $S \leq_C T$, $T \leq_C S$, *then* $S \equiv_C T$.

Lemma 2

If $S <_C T$ *are coherent trees and* $\mathcal{P}$ *is a ccc forcing, then* $\Vdash_{\mathcal{P}} S <_C T$.

Proof.

Fix a generic filter G and a club D in $V[G]$. Since $\mathcal{P}$ is ccc, we can find a club $E \in V$ such that $E \subset D$. Since $V \vDash S <_C T$, we can find a witness $\pi : S \to T$ in V for E (i.e., for any incomparable s, t in $dom(\pi)$, $\Delta_E(s,t) < \Delta_E(\pi(s), \pi(t))$. Then π is a witness for D (for some $\alpha \in E \subset D$, $\Delta(s,t) < \alpha \leq \Delta(\pi(s), \pi(t))$ and hence $\Delta_D(s,t) < \Delta_D(\pi(s), \pi(t))$). □

Relations on coherent trees

Lemma 3

If $(L, <_L)$ *is a linear order, then there is a ccc forcing such that in the forcing extension, there is a collection of coherent trees* $\mathcal{T}$, *such that* $(\mathcal{T}, <_C)$ *is order isomorphic to* $(L, <_L)$.

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Assume MA_{ω_1}. $<_C$ is a linear order on coherent trees/$\equiv_C$.

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Theorem 6

Assume MA_{ω_1}. Let $\theta, \lambda \in \{\omega, \omega_1\}$.

1. *Every $<_C$ increasing (decreasing) sequence of coherent trees of length θ has an upper (lower) bound.*
2. *(coherent trees/$\equiv_C, <_C$) has no (θ, λ) gap, i.e., if $\langle S^i : i < \theta \rangle$ is a $<_C$ increasing sequence of coherent trees, $\langle T^j : j < \lambda \rangle$ is a $<_C$ decreasing sequence of coherent trees and for any $i < \theta, j < \lambda$, $S^i <_C T^j$, then there is a coherent tree U such that $S^i <_C U <_C T^j$ for any i, j.*

The forcing axiom PFA($\mathcal{T}$)

Suppose that $\mathcal{T}$ is a collection of coherent trees and $<_{\mathcal{T}}$ is a linear order on $\mathcal{T}$. Say $\mathcal{T}$ is $<_{\mathcal{T}}$ *orderable* if there is an ω_1 preserving forcing that forces "$<_{\mathcal{T}} = <_C \restriction_{\mathcal{T}}$, i.e., $S <_C T$ for any $S <_{\mathcal{T}} T$ in $\mathcal{T}$".

Definition 3

PFA($\mathcal{T}$) asserts that $\mathcal{T}$ is a collection of coherent trees such that for some linear order $<_{\mathcal{T}}$ on $\mathcal{T}$,

1. *$\mathcal{T}$ is $<_{\mathcal{T}}$ orderable;*
2. *if $\mathcal{P}$ is a proper forcing such that $\Vdash_{\mathcal{P}}$ "$\mathcal{T}$ is $<_{\mathcal{T}}$ orderable" and $\mathcal{D}$ is a collection of ω_1 many dense subsets of $\mathcal{P}$, then there is a filter meets them all.*

Consistency of PFA($\mathcal{T}$)

Theorem 7

If there is a supercompact cardinal, $\mathcal{T}$ is a collection of coherent trees, $<_{\mathcal{T}}$ is a linear order on $\mathcal{T}$ and $\mathcal{T}$ is $<_{\mathcal{T}}$ orderable, then there is a proper forcing that forces PFA($\mathcal{T}$).

Corollary 1

If it is consistent to have a supercompact cardinal, then for any cardinal $\kappa \leq \omega_2$, it is consistent to have PFA($\mathcal{T}$) with $|\mathcal{T}| = \kappa$.

Theorem 8 (PFA($\mathcal{T}$))

1. $<_C$ *is a linear order on* $\mathcal{T}$.
2. MA_{ω_1} *holds.*

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Theorem 9 (PFA($\mathcal{T}$))

$\mathcal{T}$ *is dense in (coherent trees/*$\equiv_C, <_C$*) in the sense that for any coherent trees* $S <_C T$*, either for some* $\{S', T'\} \subset \mathcal{T}$, $S \equiv_C S'$, $T \equiv_C T'$ *or for some* $U \in \mathcal{T}$, $S <_C U <_C T$.

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Theorem 10 (PFA($\mathcal{T}$))

1. *If* $\mathcal{T}$ *has no* $<_C$ *increasing or decreasing sequence of length* ω_2*, then (coherent tree/*$\equiv_C$*,* $<_C$*) is the compactification of* $(\mathcal{T}/\equiv_C, <_C)$*, i.e., the least compact linear order containing* $\mathcal{T}$.
2. *If* $(\mathcal{T}, <_C)$ *is compact, then every coherent tree is* $\equiv_C$ *to a tree in* $\mathcal{T}$.

Consequences of PFA($\mathcal{T}$)

Proposition 1 (PFA($\mathcal{T}$))

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Proposition 2 (PFA($\mathcal{T}$))

If $(A, <_A)$ is an Aronszajn line and A has a partition tree that is tree isomorphic to a coherent tree T when restricted to a club, then $(A, <_A)$ contains either $(T, <_{lex})$ or $(-T, <_{lex})$.

Back to Aronszajn trees

Definition 4

Suppose T, U are Aronszajn trees. $T \prec_C U$ if for any club D, there is a level preserving map $\pi : T \restriction_D \to U \restriction_D$ such that for any incomparable $s, t \in T \restriction_D$, $\Delta_D(s,t) < \Delta_D(\pi(s), \pi(t))$.

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Theorem 11 (PFA($\mathcal{T}$))

Suppose U is an Aronszajn tree and T is a coherent tree. Then

1. *either there is a subtree $R \subset U$ and a club E, $R \restriction_E \leq T \restriction_E$,*
2. *or $T \prec_C U$.*

Back to Aronszajn trees

Theorem 12 (PFA($\mathcal{T}$))

Suppose that U is an Aronszajn tree. Suppose for any coherent tree T, one of the following conditions holds,

1. *for any subtree $R \subset U$, $T \prec_C R$.*
2. *for any subtree $U' \subset U$, there is a club E and a subtree $R \subset U'$, $R \restriction_E \leq T \restriction_E$.*

Then for some club D, $U \restriction_D$ is tree isomorphic to a coherent tree.

A linear order is *scattered* if it contains no isomorphic copy of $\mathbb{Q}$. For a linear order $(L, <_L)$, let linear order topology be the topology generated by $\{(a, b) : a < b \text{ in } L\}$ the collection of intervals. So a compact linear order is scattered iff its linear order topology is scattered, i.e., every nonempty subset has an isolated point.

For a compact scattered linear order $(L, <_L)$, the Cantor-Bendixson rank is defined in the following way.

$$L^{(0)} = L.$$
$$L^{(\alpha+1)} = \{x \in L^{(\alpha)} : x \text{ is not isolated in } L^{(\alpha)}\}.$$
$$L^{(\alpha)} = \bigcap_{\beta<\alpha} L^{(\beta)} \text{ if } \alpha \text{ is a limit ordinal.}$$

Then the Cantor-Bendixson rank of $(L, <_L)$ is $rank^{CB}(L) = \min\{\alpha : L^{(\alpha)} = \emptyset\}$.

Back to Aronszajn trees

Theorem 13 (PFA($\mathcal{T}$))

If $(\mathcal{T}, <_C)$ is scattered, then every Aronszajn tree contains a coherent subtree when restricted to a club.

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Corollary 2 (PFA($\mathcal{T}$))

Suppose that $(\mathcal{T}, <_C)$ is scattered. Then there is a minimal basis for uncountable linear orders and the minimal basis has size $3 + 2|\text{coherent trees}/\equiv_C|$.

Corollary 3

If it is consistent to have a supercompact cardinal, then for any cardinal n with $1 < n < \omega_2$, it is consistent to have a model in which the basis for uncountable linear orders has size $3 + 2n$.

Non-scattered case

Theorem 14 (MA_{ω_1})

If (coherent trees$/\equiv_C, <_C$) is non-scattered, then there is an Aronszajn tree containing no club restriction coherent subtree.

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For any cardinal κ, con($MA_\kappa +$ any two normal Aronszajn trees are club isomorphic).

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For any cardinal κ, con($MA_\kappa+$ any two normal Aronszajn trees are club isomorphic).

$||(club, club, \supset)|| = \min\{|\mathcal{C}| : \text{ every club contains a sub-club in } \mathcal{C}\}$.

Corollary 4 ($MA_{||(club,club,\supset)||}$)

There is an Aronszajn tree containing no coherent subtree.

Note that ccc forcing will not enlarge $||(club, club, \supset)||$. So the consistency of $MA_{||(club,club,\supset)||}$ is easily achieved.

Thank you!