Forking and Dividing at a Generic Scale

An Introduction to Kim-Independence and $NSOP_1$

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Last time

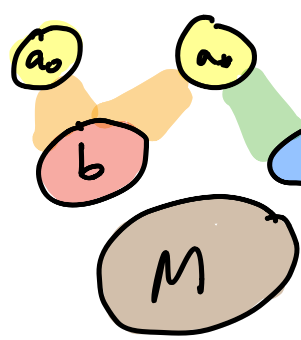
The last session was dedicated to a proof of

The Independence Theorem

Suppose T is $NSOP_1$ and $M \models T$.

Then if $a_0 \equiv_M a_1$, $a_0 \mathop{\smile\!\!\!\!\mid}^{K}_{M} b$, $a_1 \mathop{\smile\!\!\!\!\mid}^{K}_{M} c$ and $b \mathop{\smile\!\!\!\!\mid}^{K}_{M} c$, then there is some $a_* \models tp(a_0/Mb) \cup tp(a_1/Mc)$ such that

$$a_* \mathop{\smile\!\!\!\!\mid}^{K}_{M} bc.$$



The proof proceeded in 3 preliminary steps:

1. Establish a weak independence theorem:

If T is $NSOP_1$ and $M \models T$, if $a_0 \equiv_M a_1$, $a_0 \underset{M}{\mathop{\smile\!\!\!\!\!\!|}}^K b$, $a_1 \underset{M}{\mathop{\smile\!\!\!\!\!\!|}}^K c$, and $b \underset{M}{\mathop{\smile\!\!\!\!\!\!|}}^u c$, then there is $a_* \models tp(a_0/Mb) \cup tp(a_1/Mc)$ with $a_* \underset{M}{\mathop{\smile\!\!\!\!\!\!|}}^K bc$.

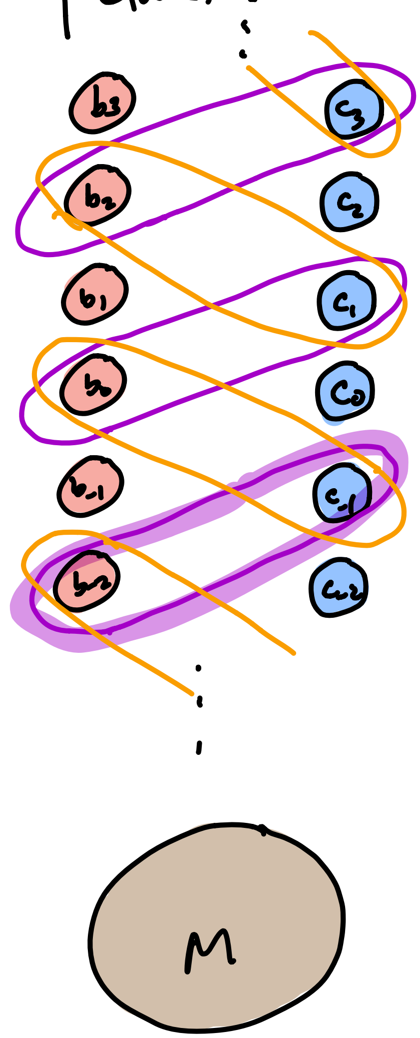
2. Prove a "consistency along trees" lemma:

If T is $NSOP_1$, $M \models T$ and $a \underset{M}{\mathop{\smile\!\!\!\!\!\!|}}^K b$, then if $(b_\eta)_{\eta \in T_\alpha}$ is a tree consisting of realizations of $tp(b/M)$, spread out over M and satisfying $b_\eta \mathop{\perp}^{u}_{M} b_{\rhd\eta}$ for all $\eta \in T_\alpha$, then there is a_* such that $a_* b_\eta \equiv_M ab$ for all $\eta \in T_\alpha$ and $a_* \mathop{\perp}^{K}_{M} (b_\eta)_{\eta \in T_\alpha}$.

3. Prove the "Zig-Zag lemma":

Suppose T is $NSOP_1$, $M \models T$, and $b \mathop{\perp}^{K}_{M} b'$. Then, for any global M-finitely satisfiable type $q \supseteq tp(b/M)$, there is a tree Morley sequence $\langle (b_i, b_i') : i < \omega \rangle$ with $(b_0, b_0') = (b, b')$ such that (1) if $i \leq j$, then $b_i b_j' \equiv_M bb'$, and (2) if $i > j$, then $b_i \models q|_{M b_j'}$.

Then we get the theorem, as in the following picture:



As a final application, we upgraded this result to get the

Strengthened Independence Theorem

Suppose T is $NSOP_1$ and $M \models T$, if $a_0 \equiv_M a_1$, $a_0 \mathop{\smile\!\!\!\!\!\!\;|}^K_M b$, $a_1 \mathop{\smile\!\!\!\!\!\!\;|}^K_M c$, and $b \mathop{\smile\!\!\!\!\!\!\;|}^K_M c$, then there is a_* such that $a_* \equiv_{Mb} a_0$, $a_* \equiv_{Mc} a_1$, and, additionally, $a_* \mathop{\smile\!\!\!\!\!\!\;|}^K_M bc$, $b \mathop{\smile\!\!\!\!\!\!\;|}^K_M a_* c$, and $c \mathop{\smile\!\!\!\!\!\!\;|}^K_M a_* b$.

Plan for 5^{th} talk

1. Give 2 further applications of the independence theorem.

2. Prove local character

3. Prove transitivity and witnessing.

$\mathop{\smile\!\!\!\!\!|}^{K}$-Morley sequences

Definition

An $\mathop{\smile\!\!\!\!\!|}^{K}$-Morley sequence over M is an M-indiscernible sequence $(a_i)_{i<\omega}$ such that $a_i \mathop{\smile\!\!\!\!\!|}^{K}_{M} a_{<i}$ for all $i<\omega$.

Observation ("Chain condition for $\mathop{\smile\!\!\!\!\!|}^{K}$-Morley sequences")

Suppose T is $NSOP_1$, $M \models T$, and $a \mathop{\smile\!\!\!\!\!|}^{K}_{M} b$. If $(b_i)_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!|}^{K}$-Morley sequence over M with $b_0 = b$, then there is $a' \equiv_{Mb} a$ such that I is Ma'-indiscernible and $a' \mathop{\smile\!\!\!\!\!|}^{K}_{M} I$.

Proof Let $p(x;b) = tp(a/Mb)$.

By induction on n, we will choose $a_n \models \bigcup_{i \leq n} p(x;b_i)$ with $a_n \mathop{\smile\!\!\!\!\!\!|}_M^K b_{\leq n}$.

To begin, we set $a_0 = a$. Then given a_n, we pick $a' \models p(x; b_{n+1})$.

Thus, we have $a_n \equiv_M a'$, $a_n \mathop{\smile\!\!\!\!\!\!|}_M^K b_{\leq n}$, $a' \mathop{\smile\!\!\!\!\!\!|}_M^K b_{n+1}$, and $b_{n+1} \mathop{\smile\!\!\!\!\!\!|}_M^K b_{\leq n}$. Then by the independence theorem, there is $a_{n+1} \models \bigcup_{i \leq n+1} p(x;b_i)$ with $a_{n+1} \mathop{\smile\!\!\!\!\!\!|}_M^K b_{\leq n+1}$.

By compactness, there is $a_* \models \bigcup_{i<\omega} p(x;b_i)$ with $a_* \mathop{\smile\!\!\!\!\!\!|}_M^K I$. Extract from I an Ma_*-indiscernible sequence $I' = \langle b_i' \mid i<\omega \rangle$.

Note that if $a_* \underset{M}{\not\mathop{\smile}\limits^{K}} I'$, then, by symmetry, $I' \underset{M}{\not\mathop{\smile}\limits^{K}} a_*$ and thus $b'_{\leq n} \underset{M}{\not\mathop{\smile}\limits^{K}} a_*$, witnessed by some $\varphi(x_0, \ldots, x_n ; a_*) \in tp(b'_{\leq n}/Ma_*)$. But because I' was extracted from I, it follows there are $i_0 < \cdots < i_n$ such that $\models \varphi(b_{i_0}, \ldots, b_{i_n} ; a_*)$, contradicting $I \underset{M}{\mathop{\smile}\limits^{K}} a_*$. So $a_* \underset{M}{\mathop{\smile}\limits^{K}} I'$ and $I' \equiv_M I$. Choose a_*' such that $a_* I' \equiv_M a_*' I$. Then $a_*' \models \bigcup_{i<\omega} p(x; b_i)$ and I is Ma_*'-indiscernible, as desired.

$\mathop{\smile}\limits^{f}$ - Morley Sequences

Recall:

Definition

An $\mathop{\smile}\limits^{f}$ - Morley sequence over A is an A-indiscernible sequence such that $a_i \mathop{\smile}\limits^{f}_{A} a_{<i}$ for all $i < \omega$.

Proposition

Suppose T is $NSOP_1$ and $M \models T$.

The following are equivalent:

(1) $\varphi(x;a)$ Kim-divides over M

(2) There is an $\mathop{\smile}\limits^{f}$-Morley sequence $\langle a_i \mid i<\omega \rangle$ over M with $a_0 = a$ such that

$\{\varphi(x;a_i) : i<\omega\}$ is inconsistent.

(3) For all $\mathop{\perp\!\!\!\!\smile}^f$-Morley sequences $\langle a_i \mid i<\omega\rangle$ over M with $a_0=a$, $\{\varphi(x;a_i) : i<\omega\}$ is inconsistent.

Proof

$(3)\Rightarrow(1)\Rightarrow(2)$ is immediate from the fact that $\mathop{\perp\!\!\!\!\smile}^i \Rightarrow \mathop{\perp\!\!\!\!\smile}^f$ and hence every Morley sequence over M in a global M-invariant type is also an $\mathop{\perp\!\!\!\!\smile}^f$-Morley sequence over M (and these always exist). Notice also $(2)\Rightarrow(1)$ by the previous proposition and $\mathop{\perp\!\!\!\!\smile}^f \Rightarrow \mathop{\perp\!\!\!\!\smile}^K$.

So now we prove (1) $\Rightarrow$ (3).

We will need one additional tool.

Definition

Suppose $M \models T$ and $\bar{a} = (a_i)_{i \in \omega}$ is an M-indiscernible sequence. A global M-invariant type $q \supseteq tp(\bar{a}/M)$ is called an indiscernible type if, whenever $\bar{a}' \models q$, $\bar{a}'$ is M-indiscernible.

Lemma

Suppose $M \models T$ and $\bar{a} = (a_i)_{i<\omega}$ is an M-indiscernible sequence.

(1) There is an indiscernible global M-invariant type $q \supseteq tp(\bar{a}/M)$.

(2) If $q \supseteq tp(\bar{a}/M)$ is an indiscernible global M-invariant type, then if $(\bar{a}_i)_{i<\omega} \models q^{\otimes\omega}|_M$, $(a_{i,j})_{i,j<\omega}$ is a mutually indiscernible array over M.

$\bar{a}_i$

i.e. $\bar{a}_i$ is $\bar{a}_{\neq i}$-indiscernible for all $i<\omega$.

Proof (1) (Sketch) Let $r \supseteq tp(\bar{a}/M)$ be a global M-invariant type and let $N \supseteq M$ be an $|M|^+$-saturated model. Let $\bar{a}' \models r|_N$ and let $\bar{b}$ be an N-indiscernible sequence extracted from $\bar{a}'$.

We claim $tp(\bar{b}/N)$ is M-invariant.

If not, there are $i_0 < \dots < i_{n-1} < \omega$ and $c \equiv_M c'$ in M such that, for some φ,

$$\models \varphi(b_{i_0}, \dots, b_{i_{n-1}}; c) \leftrightarrow \neg \varphi(b_{i_0}, \dots, b_{i_{n-1}}; c').$$

Since $\bar{b}$ was extracted from $\bar{a}'$, there are $j_0 < \dots < j_{n-1} < \omega$ such that

$\models \varphi(a'_{j_0}, \ldots, a'_{j_{n-1}}; c) \longleftrightarrow \neg\varphi(a'_{j_0}, \ldots, a'_{j_{n-1}}; c')$,

contradicting the fact that q is M-invariant. Hence $tp(\bar{b}/N)$ has a unique global M-invariant extension. Using saturation, it is easy to check this extension is indiscernible.

(2) By induction on $n < \omega$, we will show that $(\bar{a}_i)_{i \leq n}$ is a mutually indiscernible array, given M. For $n = 0$, there is nothing to show. Assume it has been established for n. As q is an indiscernible

type, we have $\bar{a}_{n+1}$ is $M\bar{a}_{\leq n}$-indiscernible. By hypothesis, if $i \leq n$, $\bar{a}_i$ is indiscernible over

$$M\bar{a}_{<i}\,\bar{a}_{i+1},\ldots,\bar{a}_n.$$

Since q is M-invariant and

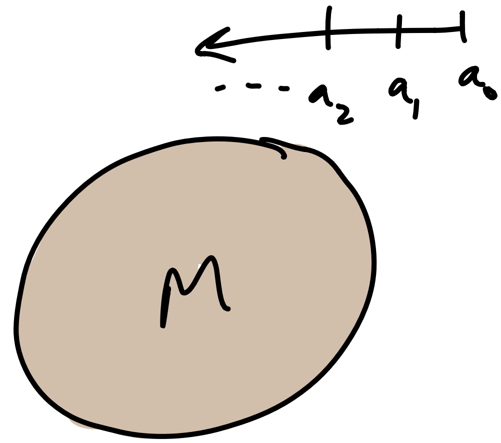
$$\bar{a}_{n+1} \models q|_{M\bar{a}_{\leq n}},$$

it follows that $\bar{a}_i$ is indiscernible over $M\bar{a}_{<i}\,\bar{a}_{i+1}\ldots\bar{a}_{n+1}$. ∎

(1) $\Longrightarrow$ (3)

We want to show that if $\varphi(x;a)$ Kim-divides over M, then for all $\mathop{\smile\!\!\!\!\!\!\,|}^{f}$-Morley sequences $I = \langle a_i \mid i<\omega \rangle$ over M with $a_0 = a$, $\{\varphi(x;a_i) : i<\omega\}$ is inconsistent.

Suppose not, assume $I = \langle a_i \mid i<\omega\rangle$ is an $\mathop{\smile\!\!\!\!\!\!\,|}^{f}$-Morley sequence over M such that $\varphi(x;a_0)$ Kim-divides over M and $\{\varphi(x;a_i) : i<\omega\}$ is consistent.



By induction on n, we will choose a sequence $(b_i)_{i\le n}$ and an elementary chain $(N_i)_{i\le n}$ such that, for all $n<\omega$,

(1) $b_0,\dots,b_n \equiv_M a_0,\dots,a_n$

(2) $M \prec N_n \prec N_{n+1}$

(3) $b_n \mathop{\smile\!\!\!\!\!\!|}^{f}_{M} N_n$

(4) $b_n \in N_{n+1}$. $\quad b_{\le n} \in N_{n+1}$

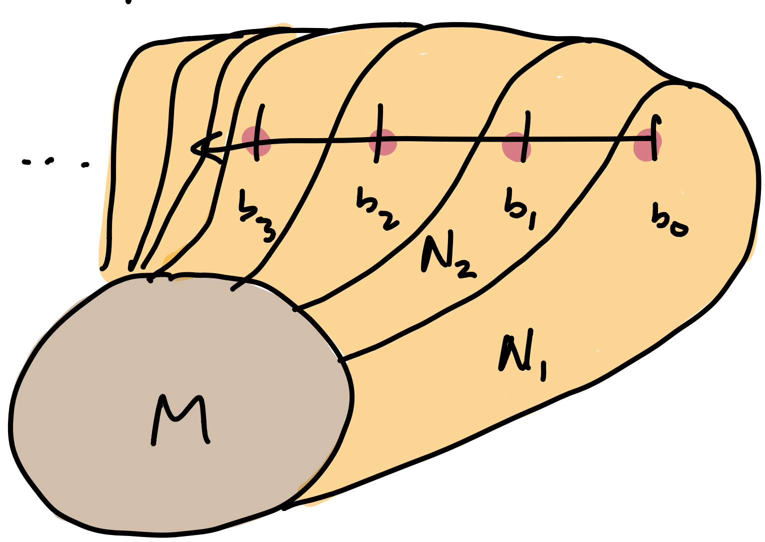
To begin, we put $b_0 = a_0$ and take

$N_0 = M$. Then, given $(b_i)_{i \leq n}$ and $(N_i)_{i \leq n}$, by (1) we can pick c such that

$$b_{\leq n} c \equiv_M a_{\leq n} a_{n+1}.$$

Then $c \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{M} b_{\leq n}$. Let N_{n+1} be a small model containing $\underline{N_n b_n}$.

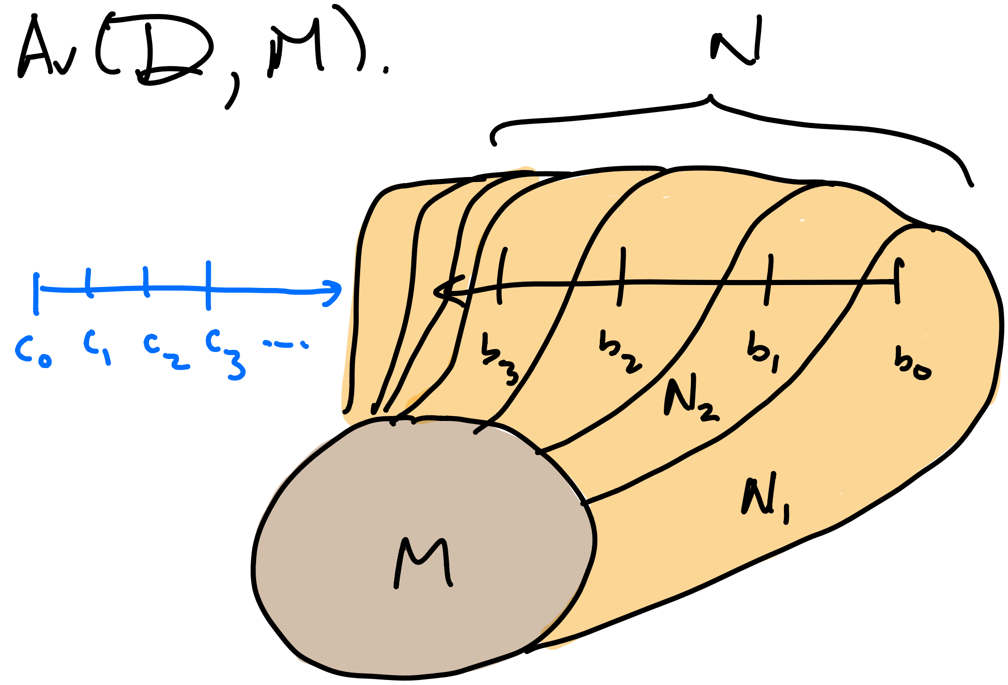
By extension, there is $b_{n+1} \equiv_{M b_{\leq n}} c$ such that $b_{n+1} \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{M} N_{n+1}$.



Then by (1), $(b_i)_{i<\omega}$ is also an $\mathop{\smile\!\!\!\!\!|}^f$-Morley sequence over M, and $\{\varphi(x;b_i) : i<\omega\}$ is consistent.

Let $N = \bigcup_{i<\omega} N_i$ and let $\mathcal{D}$ be a non-principal ultrafilter on $N^{\ell(a)}$ concentrating on $\{b_i : i<\omega\}$.

Let $(c_i)_{i<\omega}$ be a Morley sequence over N in $\mathrm{Av}(\mathcal{D}, M)$.



Claim

$$\bar{c} \mathop{\perp}\limits^{f}_{M} N.$$

Proof First, we will show $b_{\geq n} \mathop{\perp}\limits^{f}_{M} N_n$ for all $n < \omega$. By induction on $k < \omega$, we will show $b_n, \ldots, b_{n+k} \mathop{\perp}\limits^{f}_{M} N_n$.

For $k=0$, this follows by construction. Assume it for k. As $b_{\leq n+k}$ is contained in N_{n+k} and $b_{n+k+1} \mathop{\perp}\limits^{f}_{M} N_{n+k}$, we have $b_{n+k+1} \mathop{\perp}\limits^{f}_{M} N_n b_n, \ldots, b_{n+k}$. By base monotonicity, $b_{n+k+1} \mathop{\perp}\limits^{f}_{M b_n, \ldots, b_{n+k}} N_n$.

The inductive hypothesis states that

$$b_n, \dots, b_{n+k} \underset{M}{\overset{f}{\perp}} N_n ,$$

so, by left-transitivity,

$$b_n, \dots, b_{n+k+1} \underset{M}{\overset{f}{\perp}} N_n .$$

If $\varphi(x_0, \dots, x_{n-1}; m)$ is a formula over N that forks over M, then there is some K such that $m \in N_K$. The above shows

$$(\forall^\infty i_1)(\forall^\infty i_2)\dots(\forall^\infty i_{n-1})\left[\neg\varphi(b_{i_0}, \dots, b_{i_{n-1}}; m)\right]$$

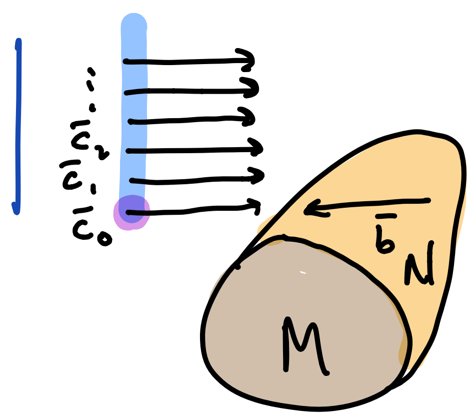
hence $\neg\varphi(c_{<n}; m)$. This shows

$$\bar{c} \underset{M}{\overset{f}{\perp}} N . \qquad \blacksquare$$

In particular, we have $N \underset{M}{\overset{K}{\perp}} \bar{c}$.

Let $q \supseteq tp(\bar{c}/M)$ be a global

M-invariant indiscernible type and let $(\bar{c}_i)_{i<\omega} \models q^{\otimes}|_M$ be a Morley sequence over M in q with $\bar{c}_0 = \bar{c}$. By $N \perp^K_M \bar{c}$, we may assume $(\bar{c}_i)_{i<\omega}$ is N-indiscernible.



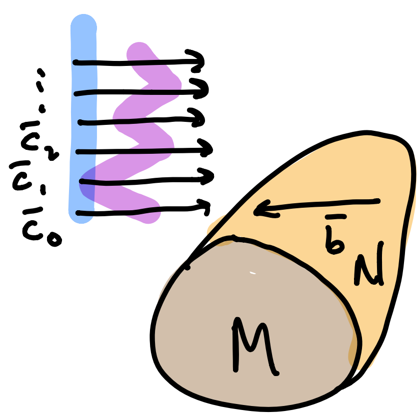
Note that because $c_{0,0} \equiv_M a$, we know $\varphi(x; c_{0,0})$ Kim-divides over M. It follows by Kim's lemma that

$$\{\varphi(x; c_{i,0}) : i<\omega\}$$

is inconsistent. But because $(\bar{c}_i)_{i<\omega}$ is a mutually indiscernible array, we know that for any function $f : \omega \to \omega$

$$(\forall) \quad \{\varphi(x; c_{i,f(i)}) : i<\omega\}$$

is inconsistent.



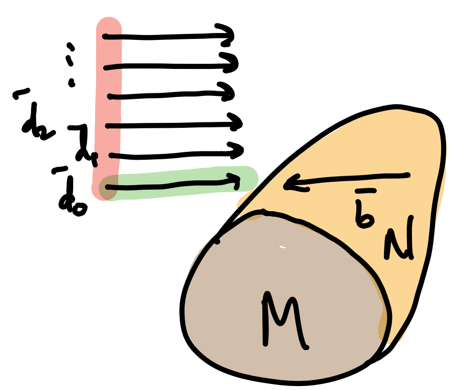
Moreover, because $(\bar{c}_i)_{i<\omega}$ is N-indiscernible and $\bar{c}_0$ is a Morley sequence in an N-finitely satisfiable type, $\bar{c}_i$ is a Morley sequence in an N-finitely satisfiable type. Moreover,

$$\{\varphi(x; c_{i,j}) : j<\omega\}$$

is consistent for all $i<\omega$.

Let $(d_{i,j})_{i,j<\omega}$ be a mutually indiscernible array over N extracted from $(c_{i,j})_{i,j<\omega}$.

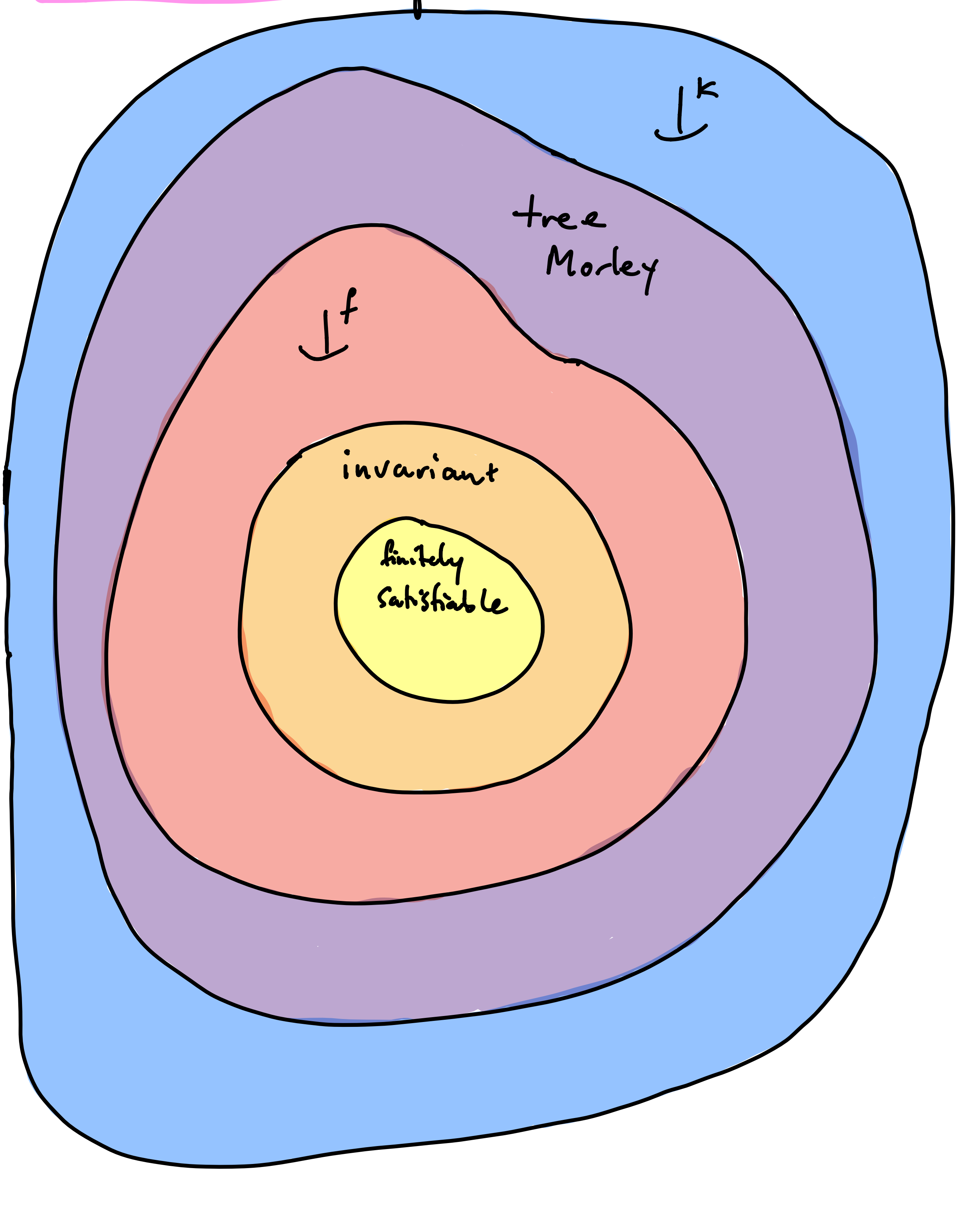
By $(*)$, $\{\varphi(x; d_{i,0}) : i<\omega\}$ is inconsistent.



Likewise, $\{\varphi(x; d_{0,j}) : j<\omega\}$ is consistent and $\bar{d}_0$ is a Morley sequence in a global N-finitely satisfiable type, so $\varphi(x; d_{0,0})$

does not Kim-divide over N. Moreover, $\bar{d}_i$ is $N\bar{d}_{<i}$-indiscernible so $\bar{d}_{<i} \perp^K_N d_{i,0}$ by Kim's lemma. It follows that $(d_{i,0})_{i<\omega}$ is an $\perp^K$-Morley sequence over N. Since $\varphi(x; d_{0,0})$ does not Kim-divide over N, $\{\varphi(x; d_{i,0}) : i<\omega\}$ is consistent (by the chain condition from before). That's a contradiction. ∎

Generic Sequences



Local Character

Recall local character for simple theories:

(*) for all sets A and finite tuples a, there is $B \subseteq A$ with $|B| \leq |T|$ such that $a \mathop{\downarrow}^{f}_{B} A$.

This asserts the existence of one set (of size $\leq |T|$) over which $tp(a/A)$ does not fork — but it immediately, by base monotonicity implies the existence of many, since if $a \mathop{\downarrow}^{f}_{B} A$, then $a \mathop{\downarrow}^{f}_{C} A$ for all sets C with $B \subseteq C \subseteq A$. This is what is actually used in applications.

The Generalized Club Filter

Definition

Suppose X is a set and κ is a cardinal.

We write $[X]^\kappa$ for $\{Y \subseteq X : |Y| = \kappa\}$.

A subset $C \subseteq [X]^\kappa$ is called a club if it is

(1) closed: If $\langle Y_i : i < \alpha \leq \kappa \rangle$ is a chain in C — i.e. $Y_i \subseteq Y_{i+1}$ and $Y_i \in C$ for all $i < \alpha$ — then $\bigcup_{i<\alpha} Y_i \in C$.

(2) unbounded: If $Z \in [X]^\kappa$, there is $Y \in C$ with $Z \subseteq Y$.

The (generalized) club filter on $[X]^\kappa$ is the filter generated by the clubs.

Example

Suppose M is an L-structure.

$$\{ N \preccurlyeq M \mid |N| = |L| \}$$

is a club in $[M]^{|T|}$.

Proof Trivial if $|M| \leq |T|$. In general

(a) $\{ N \preccurlyeq M \mid |N| = |T| \}$ is closed because a union of an elementary chain is an elementary extension.

(b) $\{ N \preccurlyeq M \mid |N| = |T| \}$ is unbounded by downward Lowenheim-Skolem.

Key Facts

(1) If $\gamma < \kappa$ and $(C_i)_{i<\gamma}$ is a sequence of clubs in $[X]^\kappa$, then

$$\bigcap_{i<\gamma} C_i$$

is club in $[X]^\kappa$

(2) The club filter is also closed under diagonal intersection — that is, if $(C_x)_{x\in X}$ is a family of clubs,

$$\underset{x\in X}{\Delta} C_x := \{ Y \in [X]^\kappa \mid Y \in \bigcap_{x\in Y} C_y \}$$ is club.

Definition

A set $S \subseteq [X]^\kappa$ is called stationary if

$S \cap C \neq \emptyset$ for every club $C \subseteq [X]^\kappa$.

Analogy

measure 1	:	club
positive measure	:	stationary

Our aim now is to prove the following

Theorem

The following are equivalent:

(1) T is $NSOP_1$

(2) If $M \models T$, for every $p \in S_x(M)$,

$\{N \prec M \mid |N| = |T| \text{ and } p \text{ does not Kim-fork over } N\}$

is stationary.

(3) If $M \models T$, for every $p \in S_x(M)$,

$\{N \prec M \mid |N| = |T|$ and p does not Kim-fork over $N\}$

contains a club.

$(2) \Rightarrow (1)$

We will assume T has SOP_1 and construct a model M and $p \in S_x(M)$ such that

$$\{N \prec M \mid |N| = |T| \text{ and } p \text{ Kim-forks over } N\}$$

contains a club. This implies

$$\{N \prec M \mid |N| = |T| \text{ and } p \text{ does not Kim-fork over } N\}$$

is not stationary.

Fix a Skolemization T^{Sk} of T, with $|T^{Sk}| = |T|$.

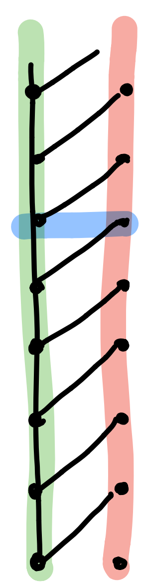
As T has SOP_1, there is some L-formula $\varphi(x;y)$ and an array $(c_{i,0}, c_{i,1})_{i<|T|^+}$ such that

(1) $\{\varphi(x;c_{i,0}) : i < |T|^+\}$ is consistent.

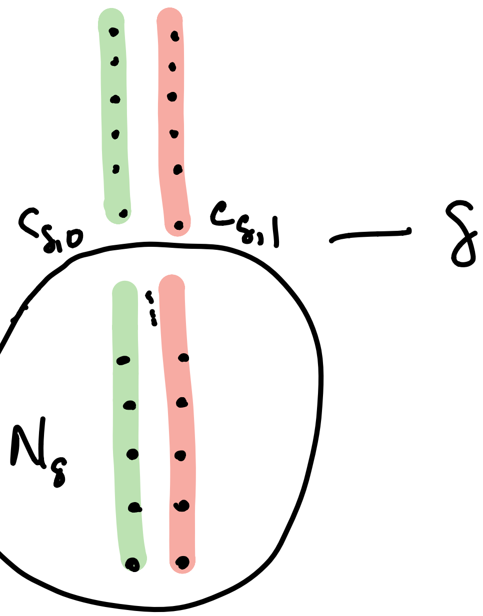
(2) $\{\varphi(x;c_{i,1}) : i < |T|^+\}$ is 2-inconsistent.

(3) $c_{i,0} \equiv^{L^{sk}}_{\bar{c}_{<i}} c_{i,1}$ for all $i < |T|^+$.

(4) $(\bar{c}_i)_{i<|T|^+}$ is L^{sk}-indiscernible.



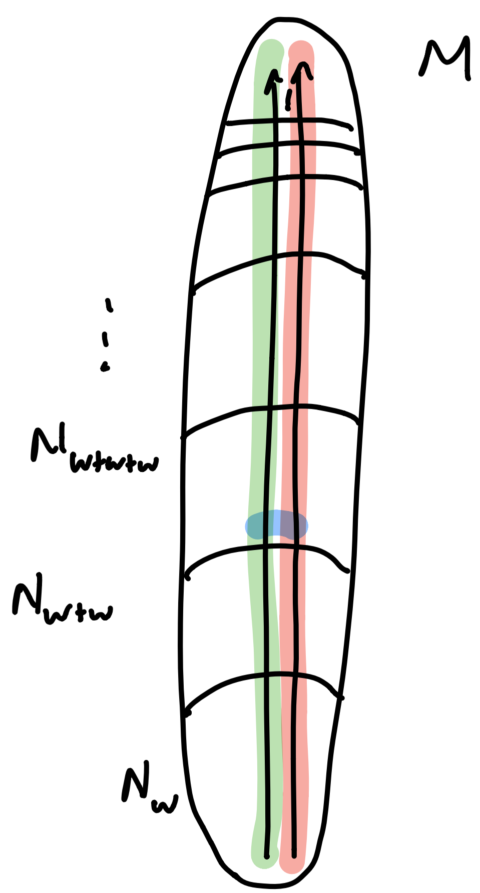
Let $M = \mathrm{dcl}_{L^{sk}}(\bar{c}_{<|T|^+}) \upharpoonright L$.



Let $N_\delta = Sk(\bar{c}_{<\delta})$. If δ is a limit ordinal, then $(c_{i,1})_{i \geq \delta}$ is a Morley sequence in a global N_δ-finitely satisfiable type extending $tp_L(c_{\delta,1}/N_\delta)$, by L^{sk}-indiscernibility of $(\bar{c}_i)_{i<|T|^+}$. But because $c_{\delta,0} \equiv^{L^{sk}}_{\bar{c}_{<\delta}} c_{\delta,1}$, we know $c_{\delta,0} \equiv^{L}_{N_\delta} c_{\delta,1}$.

Hence $\varphi(x'; c_{\delta,0})$ Kim-divides over N_δ,

as $\{\varphi(x'; c_{i,1}) : i \geq \delta\}$ is 2-inconsistent.



Let p be any completion over M of $\{\varphi(x'; c_{\delta,0}) : \delta < |T|^+\}$.

As $\{ \mathrm{dcl}_{\mathrm{Sk}}(\bar{c}_{<\delta}) \mid \delta \in \lim(|T|^+) \}$ is club in $[M]^{|T|}$, we have completed the proof.

Heirs

Definition

Suppose $M \prec N$ and $q \in S_x(N)$.

We say q is an heir of $q|_M$ if,

for every formula $\varphi(x;y) \in L(M)$, if $\varphi(x;n) \in q$,

then there is some $m \in M$ with $\varphi(x;m) \in q|_M$.

Equivalently, q is an heir of $q|_M$ if,

for some (equivalently, all) $a \models q$,

$$N \mathop{\perp\!\!\!\!\perp}^{u}_{M} a.$$

Lemma

Suppose N is a model and $p \in S_x(N)$.

Then $\{ M \preceq N \mid |M| = |T| \text{ and } p \text{ is an heir of } p \upharpoonright M \}$

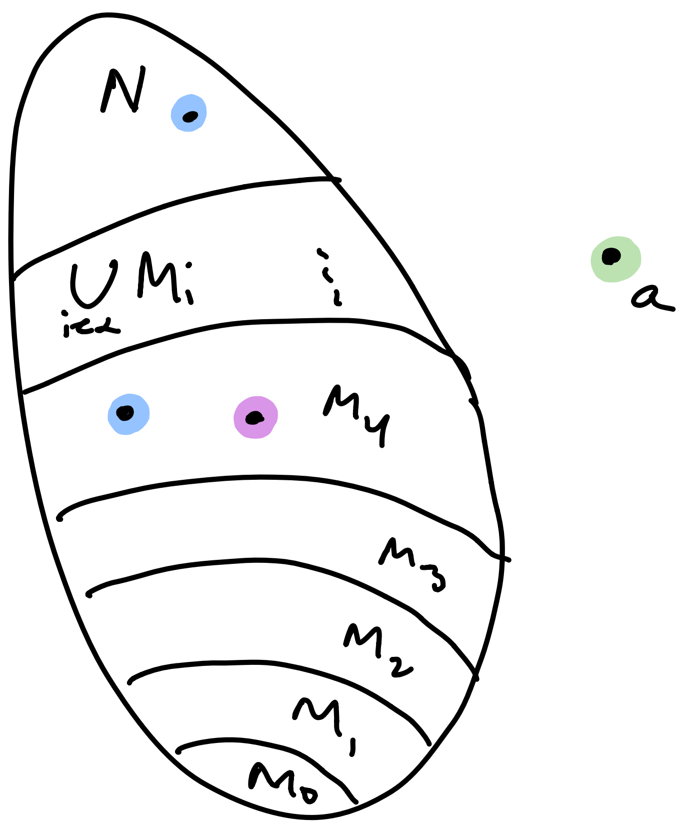
is a club of $[N]^{|T|}$.

Proof

- closed: Fix $a \models p$ and suppose $(M_i)_{i<\alpha}$ is a chain of elementary submodels of N such that $N \mathop{\smile\!\!\!\!\!\mid}^{u}_{M_i} a$ for all $i<\alpha$. We may assume α is a limit ordinal. Then if $\varphi(z;a) \in tp(N/\bigcup_{i<\alpha} M_i\, a)$, then the parameters of φ all come from M_i a

for some $i < \alpha$, hence the formula in

satisfied in M_i.



- unbounded: Expand T to make p definable — that is, for each formula $\varphi(x;y)$, add a $|y|$-ary relation symbol $R_\varphi(y)$, interpreted so that
$$R_\varphi^N = \{c \in N^{|y|} \mid \varphi(x;c) \in p\}.$$
Let L_p be this larger language. Note that $|L_p| = |L|$ so, by downward Lowenheim-Skolem, if $X \subseteq N$ has size $|T|$, there is an L_p-elementary substructure $M \preccurlyeq N$ such that $X \subseteq M$ and

$|M| = |T|$. It is easy to check that $M \preceq^{L_p} N$ implies p is an heir of $p|M$ so M is in our set. $\blacksquare$

(1) $\Longrightarrow$ (3) (P. Simon)

Suppose T is $NSOP_1$, $M \models T$, and $p \in S_x(M)$. We want to show

$$\{N \preceq M \mid |N| = |T| \text{ and } p \text{ does not Kim-fork over } N\}$$

contains a club.

Let $a \models p$. Then if $N \preceq M$ and p is an heir of $p|_N$, we have $M \mathop{\smile\!\!\!\!\!|}^{u}_{N} a$ and hence $a \mathop{\smile\!\!\!\!\!|}^{K}_{N} M$ by symmetry. The previous lemma, then, shows that

$$\{N \preceq M \mid |N| = |T| \text{ and } p \text{ does not Kim-fork over } N\}$$

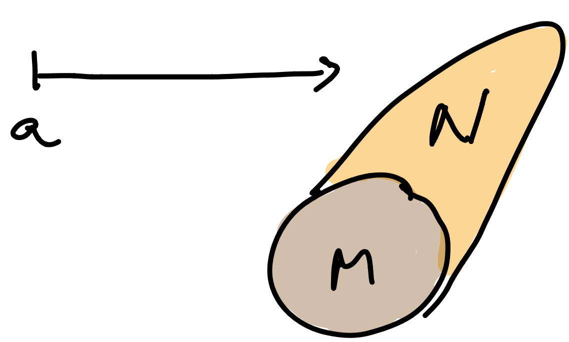
contains a club. $\blacksquare$

Towards transitivity

Lemma

Suppose T is $NSOP_1$, $M \preceq N \models T$ and $a \underset{M}{\perp\!\!\!\perp}^K N$.

Then there is a sequence $I = (a_i)_{i<\omega}$ such that simultaneously a tree Morley sequence over N and $\perp\!\!\!\perp^K$-Morley over M, with $a_0 = a$.



Proof

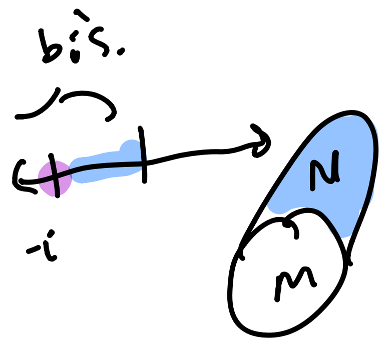
Let $\langle a_i : i \in \mathbb{Z} \rangle$ be a Morley sequence in a global N-finitely satisfiable type with $a_0 = a$. Then this sequence is a tree Morley sequence over N. Let $\langle b_i : i < \omega \rangle$ be defined by $b_i = a_{-i}$ for all $i < \omega$. Then $\bar{b}$ is a tree Morley sequence over N.

We claim additionally that

$$b_i \mathop{\perp}_{M}^{K} b_{<i} N \quad \text{for all } i < \omega.$$

By symmetry, it suffices to show

$$b_{<i} N \underset{M}{\overset{K}{\perp}} b_i ;$$



or, equivalently,

$$a_{>-i} N \underset{M}{\overset{K}{\perp}} a_{-i} .$$

So suppose $\varphi(a_{>-i}, n; a_{-i})$.

As $a_{>-i} \underset{N}{\overset{u}{\perp}} a_i$, there is some $n' \in N$ such that

$$\models \varphi(n', n; a_{-i}),$$

and as $a \equiv_N a_{-i}$, it follows that $\varphi(z, y; a_{-i})$ Kim-divides over M if and only if $\varphi(z, y; a)$ Kim-divides over M, and also $\models \varphi(n; n; a_{-i})$ if and only if $\models \varphi(n; n; a)$.

As $N \mathop{\smile\!\!\!\!\!\!\mid}^K_M a$, by symmetry, we conclude $\varphi(z, y; a_{-i})$ does not Kim-fork over M.

This shows $b_i \mathop{\smile\!\!\!\!\!\!\mid}^K_M b_{<i} N$. ∎