Forking and Dividing at a Generic Scale

An Introduction to Kim-Independence and $NSOP_1$

Nicholas Ramsey

UCLA

Last Time

(1) We introduced s-indiscernible trees, which are generalized indiscernibles indexed by trees, which are viewed as L_s-structures, where $L_s = \{\trianglelefteq, <_{lex}, \wedge, (P_\alpha)_{\alpha \in \omega}\}$ (We also use this name for T_α-indexed indiscernibles, where T_α is viewed as an $L_{s,\alpha}$ $\left(= \{\trianglelefteq, <_{lex}, \wedge, (P_\beta)_{\beta \in nl(\alpha)}\}\right)$-structure).

(2) We introduced the trees T_α as the natural index models for the inductive construction of Morley

trees. We defined a tree Morley sequence to be the all 0's path in some Morley tree

(3) We proved Kim's lemma for tree Morley sequences: for $NSOP_1$ T and $M \models T$, $\varphi(x;a)$ Kim-divides over M if and only if $\varphi(x;a)$ divides with respect to every tree Morley sequence over M, starting with a.

(4) We also showed if T is $NSOP_1$ and $M \models T$, then $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$ implies there is an Mb-indiscernible

tree Morley sequence over M, starting with a. This combined with Kim's lemma to yield the following:

Theorem

The following are equivalent:

(1) T is $NSOP_1$.

(2) Symmetry: If $M \models T$, then

$$a \underset{M}{\downarrow}^{K} b \iff b \underset{M}{\downarrow}^{K} a.$$

Recalling definitions for T_α.

$$T_\alpha = \left\{ f : [\beta, \alpha) \to \omega \;\middle|\; \begin{array}{l} \beta \in n\ell(\alpha+1), \\ f \text{ of finite} \\ \text{support} \end{array} \right\}$$

- Canonical inclusions: if $\alpha < \beta$, then

  $$\iota_{\alpha\beta}(\eta) = \eta \cup \{(\gamma, 0) \mid \gamma \in \beta \setminus \alpha\}$$

  for all $\eta \in T_\alpha$.

- Concatenation: Suppose $\eta \in T_\alpha$ and $i < \omega$.

  (i) If $\operatorname{dom}(\eta) = [\beta+1, \alpha)$, then

  $$\eta^\frown\langle i\rangle = \eta \cup \{(\beta, i)\}.$$

  (ii) $\langle i\rangle^\frown\eta \in T_{\alpha+1}$ is defined by

  $$\eta \cup \{(\alpha, i)\}.$$

- **All 0's path**: If $\beta \in n\ell(\alpha)$, we define $\zeta_\beta \in T_\alpha$ to be the function with domain $[\beta, \alpha)$ which is identically zero. We refer to $(\zeta_\beta)_{\beta \in n\ell(\alpha)}$ as the **all 0's path**.
- **Restriction**: If $\nu \subseteq n\ell(\alpha)$, we write $T_\alpha \restriction \nu = \left\{ \eta \in T_\alpha \,\middle|\, \begin{array}{l} \min(\mathrm{dom}(\eta)) \in \nu, \\ \beta \in \mathrm{dom}(\eta) \setminus \nu \Rightarrow \eta(\beta) = 0 \end{array} \right\}$

Morley Trees

Definition

We say $(a_\eta)_{\eta \in T_\alpha}$ is a Morley tree over M if

(i) $(a_\eta)_{\eta \in T_\alpha}$ is s-indiscernible over M.

(ii) $(a_\eta)_{\eta \in T_\alpha}$ is spread out over M; i.e. if $\mathrm{dom}(\eta) \neq \alpha$ then $(a_{\trianglerighteq \eta^\frown \langle i \rangle})_{i \in \omega}$ is Morley over M in some global M-invariant type.

(iii) If $\nu, \nu' \in [n\ell(\alpha)]^{<\omega}$ and $|\nu| = |\nu'|$, then
$$(a_\eta)_{\eta \in T_\alpha \restriction \nu} \equiv_M (a_\eta)_{\eta \in T_\alpha \restriction \nu'}.$$

Tree Morley Sequences

Definition

A tree Morley sequence over M is a sequence $(a_i)_{i<\omega}$ such that, for some Morley tree over M, $(a_\eta)_{\eta\in T_\omega}$, $a_i = a_{\zeta_i}$ for all $i<\omega$.

More generally, if I is an infinite linear order, we will say $(b_i)_{i\in I}$ is a tree Morley sequence over M if there is a tree Morley sequence $(a_i)_{i<\omega}$ such that, for all $i_0<\dots<i_{n-1}\in I$,

$$(b_{i_0},\dots b_{i_{n-1}}) \equiv_M (a_0,\dots,a_{n-1}).$$

The Chain Condition

Fact

If $(a_i)_{i<\omega}$ is a tree Morley sequence over M, then $\langle (a_{k\cdot i}, \ldots, a_{k\cdot i+k-1}) : i<\omega \rangle$ is also a tree Morley sequence over M, for any $1 \leq k < \omega$.

Corollary

If T is $NSOP_1$, $M \models T$, and $a \mathop{\perp}^{K}_{M} b$, then, given any tree Morley sequence $I = \langle b_i \mid i<\omega \rangle$ over M with $b_0 = b$, there is $a' \equiv_{Mb} a$ such that

I is Ma'-indiscernible and $a' \mathop{\perp}\limits_M^K I$ (or equivalently, there is $I' \equiv_{Mb} I$ such that I' is Ma-indiscernible and $a \mathop{\perp}\limits_M I'$).

Proof As $a \mathop{\perp}\limits_M^K b$ and tree Morley sequences witness Kim-dividing, there is $a' \equiv_{Mb} a$ such that I is Ma'-indiscernible. Then, for all $1 \leq k < \omega$, $\langle (b_{k \cdot i}, b_{k \cdot i+1}, \ldots, b_{k \cdot i+k-1}) : i < \omega \rangle$ is Ma'-indiscernible. Since this sequence is a tree Morley sequence over M, this shows $a' \mathop{\perp}\limits_M^K b_{<k}$ for all k. $\boxed{}$

A Weak Independence Theorem

Proposition

Assume T is $NSOP_1$ and $M \models T$.

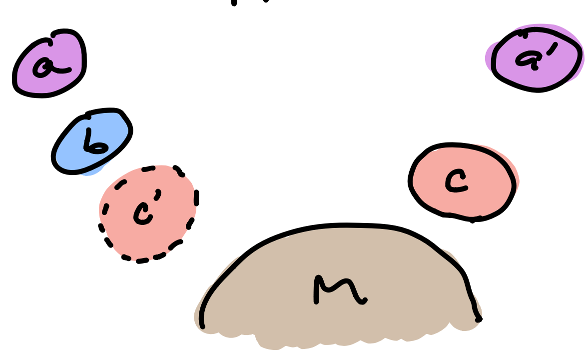
If $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M c$, and $b \mathop{\smile\!\!\!\!\!\!\;\mid}^u_M c$, then there is a_* with

$a_* \equiv_{Mb} a$, $a_* \equiv_{Mc} a'$, and $a_* \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M bc$.

Proof We start with the following

Claim There is c' such that

$$ac' \equiv_M a'c, \text{ and } a \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M bc'.$$



Proof of Claim By symmetry, it suffices to find c' such that $ac' \equiv_M a'c$ and $bc' \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M a$. Let $p(x;y) = tp(c;a'/M)$.

We must show

$$p(x;a) \cup \{\neg\varphi(x,b;a) \mid \begin{array}{l}\varphi(x,z;a) \text{ Kim-forks} \\ \text{over } M\end{array}\}$$

is consistent. If not, as Kim-forking and Kim-dividing are the same,

$$p(x;a) \vdash \varphi(x,b;a)$$

for some $\varphi(x,z;a)$ that Kim-divides over M. Let $I = (a_i)_{i\in\omega}$ be an Mb-indiscernible tree Morley sequence over M with $a_0 = a$ (possible as $a \mathop{\perp\!\!\!\!\smile}^{K}_{M} b$). As $c \mathop{\perp\!\!\!\!\smile}^{K}_{M} a'$ and $a' \equiv_M a$, we know $p(x;a)$ does not Kim-fork over M, and, hence, $\bigcup_{i<\omega} p(x;a_i)$ is consistent. But $\bigcup_{i\in\omega} p(x;a_i) \vdash \{\varphi(x,b;a_i) : i<\omega\}$, and $\{\varphi(x,z;a_i) : i<\omega\}$ is inconsistent, by Kim's lemma, since $\varphi(x,y;a_0)$ Kim-divides over M. Contradiction.

Hence this set of formulas is consistent, and we may take c' to be any realization. $\blacksquare$

Now as $b \mathop{\smile\!\!\!\!\!\mid}^{u}_{M} c$, by left-extension of $\mathop{\smile\!\!\!\!\!\mid}^{u}$, we may assume $bc' \mathop{\smile\!\!\!\!\!\mid}^{u}_{M} c$.

Now we choose b' such that

$$bc' \equiv_M b'c.$$

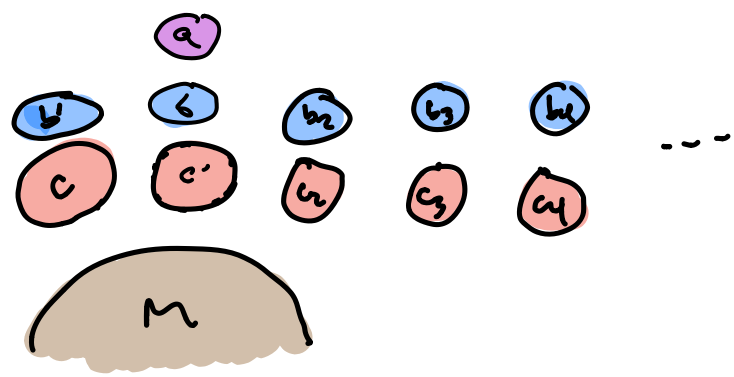
By (right) extension for $\mathop{\smile\!\!\!\!\!\mid}^{u}$, we may assume $bc' \mathop{\smile\!\!\!\!\!\mid}^{u}_{M} b'c$.

Let q be a global M-finitely satisfiable type with

$$(b,c') \models q|_{Mb'c}.$$

Then there is a Morley sequence $I = (b_i, c_i)_{i\in\omega}$ over M in q

with $(b_0, c_0) = (b', c)$ and
$(b_1, c_1) = (b, c')$



As $a \underset{M}{\mathop{\perp}\limits^{K}} bc'$, there is $a_* \equiv_{Mbc'} a$ such that I is Ma_*-indiscernible and $a_* \underset{M}{\mathop{\perp}\limits^{K}} I$ by the chain condition.

Then, in particular, $a_* \underset{M}{\mathop{\perp}\limits^{K}} bc$.

We know $a_* \equiv_{Mb} a$ and $a_* \equiv_{Mc'} a$, hence, by indiscernibility, $a_* c \equiv_M a_* c' \equiv_M ac' \equiv_M a'c$. ∎

Consistency Along Trees

Lemma

Suppose T is $NSOP_1$, $M \models T$, and $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$. If $p(x;y) = tp(a;b/M)$ and $(b_\eta)_{\eta \in T_\alpha}$ is a tree, spread out over M, such that, for some global M-finitely satisfiable type q extending $tp(b/M)$, $b_\eta \models q|_{Mb_{\rhd \eta}}$ for all $\eta \in T_\alpha$, then

$$\bigcup_{\eta \in T_\alpha} p(x;b_\eta)$$

is consistent and non-Kim-forking over M.

Proof The proof is by induction on α. The case of $\alpha = 0$ is immediate by the assumption that $a \mathop{\smile\!\!\!\!\!\!\,|}^{K}_{M} b$.

Suppose it has been shown for all such trees indexed by T_α, and consider $(b_\eta)_{\eta \in T_{\alpha+1}}$.

As $(b_\eta)_{\eta \in T_{\alpha+1}}$ is spread out over M, we know $\langle b_{\trianglerighteq \langle i \rangle} \mid i < \omega \rangle$ is a Morley sequence over M in some global M-invariant type.

The tree $(b'_\eta)_{\eta \in T_\alpha}$ defined by $b'_\eta = b_{\langle 0 \rangle^\frown \eta}$ for all $\eta \in T_\alpha$ is

$= b_{\iota_{2\to 1}(\eta)}$

also spread out

over M, so the inductive hypothesis

gives $\bigcup_{\eta \in T_2} p(x; b'_\eta) = \bigcup_{\eta \in T_2} p(x; b_{\langle 0 \rangle \frown \eta})$

is consistent and non-Kim-forking over M.

Therefore, $\bigcup_{i<\omega} \bigcup_{\eta \in T_2} p(x; b_{\langle i \rangle \frown \eta})$

is consistent and non-Kim-forking.

Let a_0 be a realization. Let

$a_1 \models p(x; b_\emptyset)$. Then we have

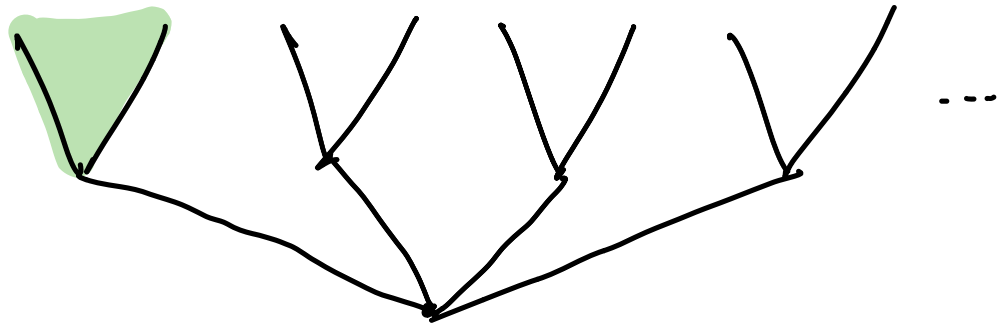
$a_0 \equiv_M a_1$, $a_0 \mathop{\smile\!\!\!\!\!\!|}^K_M b_{\rhd \emptyset}$, $a_1 \mathop{\smile\!\!\!\!\!\!|}^K_M b_\emptyset$

and $b_\emptyset \mathop{\smile\!\!\!\!\!\!|}^u_M b_{\rhd \emptyset}$, hence there is

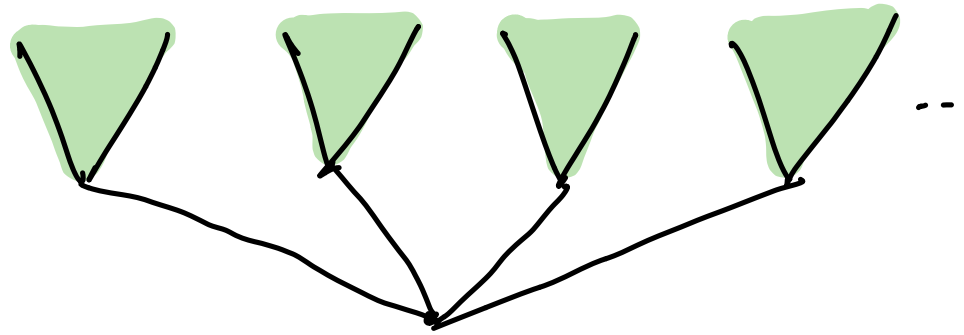
$a_* \models \bigcup_{\eta \in T_{2+1}} p(x; b_\eta)$ with $a_* \mathop{\smile\!\!\!\!\!\!|}^K_M (b_\eta)_{\eta \in T_2}$

by the weak independence theorem.

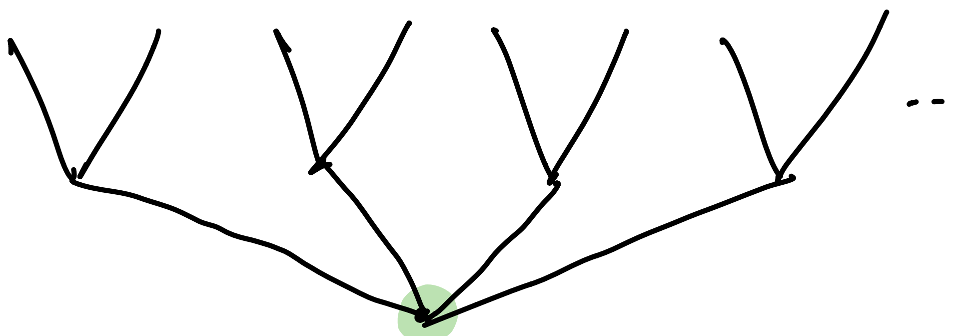
1



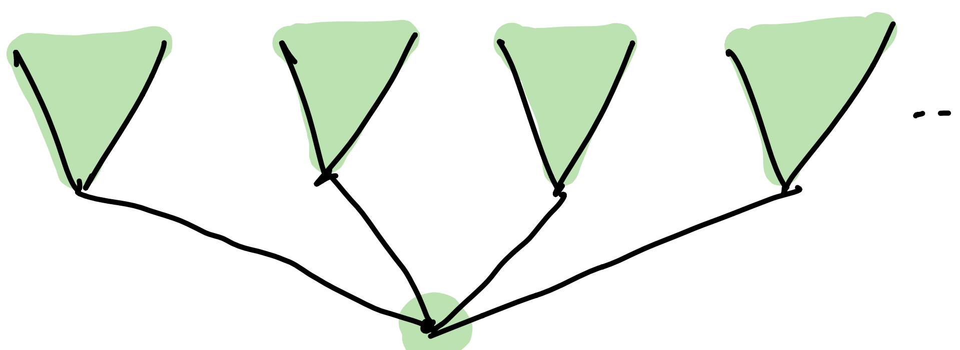
2



3



4



Remark

For δ limit, follows by finite character + compactness. $\blacksquare$

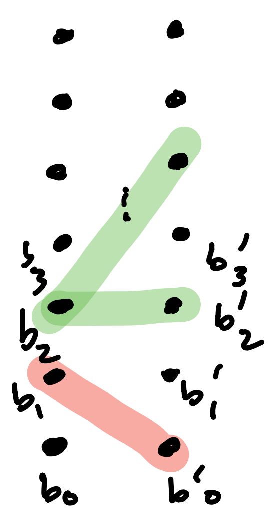
The hypothesis that $b_\eta \models q|_{M b_{\rhd \eta}}$ for all $\eta \in T_\alpha$ is needed only so that we may apply the weak independence theorem. Once we know the full independence theorem, we could conclude $\bigcup_{\eta \in T_\alpha} p(x; b_\eta)$ is Kim-non-forking over M, instead assuming that $(b_\eta)_{\eta \in T_\alpha}$ is spread out and s-indiscernible over M, since this implies $b_\eta \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} b_{\rhd \eta}$ for all $\eta \in T_\alpha$.

Zig-zag lemma

Suppose T is $NSOP_1$, $M \models T$ and $b \mathop{\smile\!\!\!\!\!\!\;|}^{K}_{M} b'$.
Then for any global M-finitely satisfiable type $q \supseteq tp(b/M)$, there is a tree Morley sequence $(b_i, b_i')_{i\in\omega}$ over M with $(b_0, b_0') = (b, b')$ such that

(1) If $i \le j$ then $b_i b_j' \equiv_M bb'$

(2) If $i > j$, then $b_i \models q|_{Mb_j'}$.



Proof Fix $q \supseteq tp(b/M)$ and let $p(x;b) = tp(b';b/M)$. By recursion on α, we will construct $(c^\alpha_\eta, d^\alpha_\eta)_{\eta \in T_\alpha}$ such that

(1) If $\eta \in T_\alpha$, then

$$c^\alpha_\eta \models q|_{c^\alpha_{\rhd\eta} d^\alpha_{\rhd\eta} M}$$

(2) If $\eta \in T_\alpha$, then

$$d^\alpha_\eta \models \bigcup_{\nu \trianglerighteq \eta} p(x; c^\alpha_\nu)$$

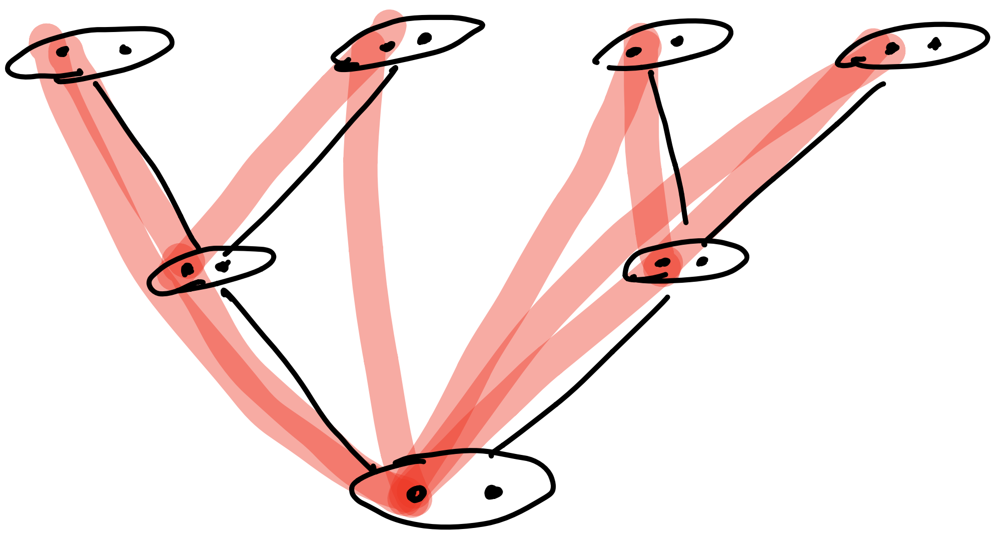
(3) $(c^\alpha_\eta, d^\alpha_\eta)_{\eta \in T_\alpha}$ is s-indiscernible and spread out over M.

(4) If $\beta < \alpha$, then

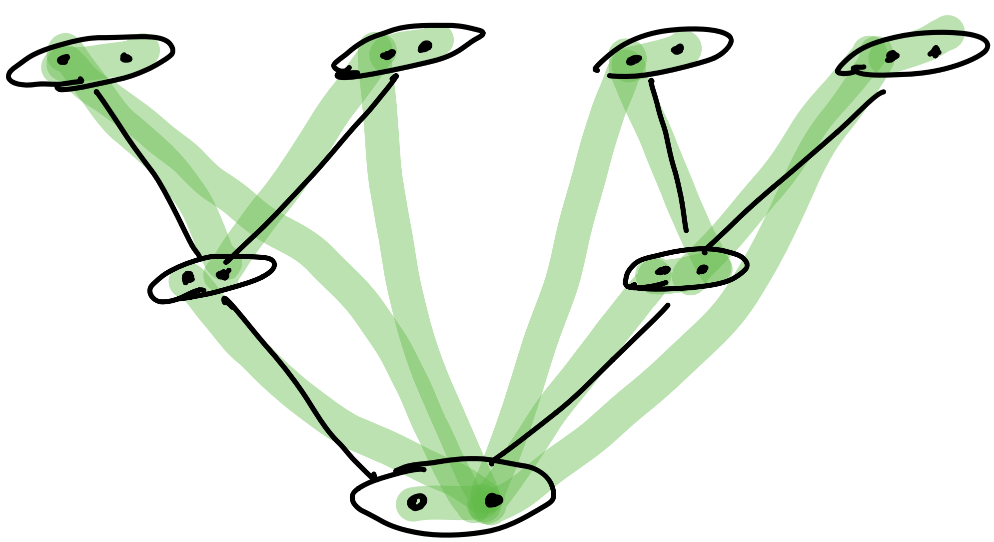
$$\left(c^{\alpha}_{\iota_{\beta\alpha}(\eta)}, d^{\alpha}_{\iota_{\beta\alpha}(\eta)}\right) = \left(c^{\beta}_{\eta}, d^{\beta}_{\eta}\right)$$

for all $\eta \in T_{\beta}$.

1



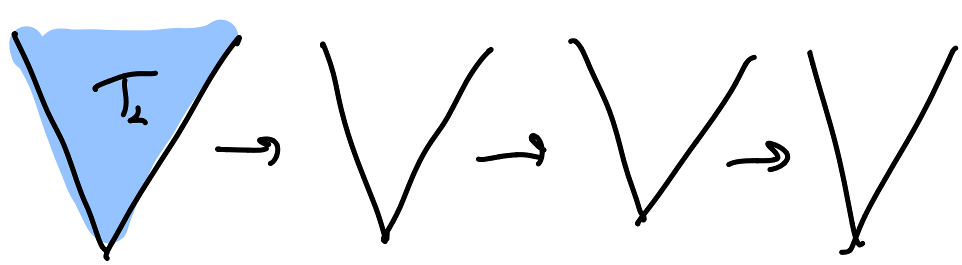
2



To begin, we define $(c^0_\eta, d^0_\eta)_{\eta \in T_0}$ by setting $(c^0_\emptyset, d^0_\emptyset) = (b, b')$.

Now suppose we are given $(c^\beta_\eta, d^\beta_\eta)_{\eta \in T_\beta}$ for all $\beta \leq \alpha$.

Let $\langle (c^\alpha_{\eta,i}, d^\alpha_{\eta,i})_{\eta \in T_\alpha} \mid i < \omega \rangle$ be a Morley sequence over M in some global M-invariant type extending $tp((c^\alpha_\eta, d^\alpha_\eta)_{\eta \in T_\alpha} / M)$, with $(c^\alpha_{\eta,0}, d^\alpha_{\eta,0})_{\eta \in T_\alpha} = (c^\alpha_\eta, d^\alpha_\eta)_{\eta \in T_\alpha}$.



Let $c_* \models q|_{M(c^\alpha_{\eta,i}, d^\alpha_{\eta,i})_{\eta \in T_\alpha, i<\omega}}$.

Define $(e_\eta)_{\eta \in T_{\alpha+1}}$ by

$$e_\emptyset = c_*$$

$$e_{\langle i \rangle ^\frown \eta} = c^\alpha_{\eta,i} \quad \text{for all } \eta \in T_\alpha, \; i<\omega.$$

Then $(e_\eta)_{\eta \in T_\alpha}$ is spread out over M and $e_\emptyset \models q|_{M(e_{\rhd\emptyset})(d^\alpha_{\eta,i})_{i<\omega, \eta \in T_\alpha}}$ $\eta \in T_{\alpha+1}$. Hence, by the previous lemma, $\bigcup_{\eta \in T_{\alpha+1}} p(x; e_\eta)$ is Kim non-forking over M. Let d_* be a realization.

$e_\emptyset \models q|_{M e_{\eta \not\models \emptyset}}$

Now define $(f_\eta)_{\eta \in T_{\alpha+1}}$ by

$$f_\emptyset = d_*$$
$$f_{\langle i\rangle ^\frown \eta} = \underline{d^\alpha_{\eta,i}} \quad \text{for all } \eta \in T_\alpha,\ i<\omega.$$

Then the tree $(e_\eta, f_\eta)_{\eta \in T_{\alpha+1}}$ satisfies the desired conditions for $(c^{\alpha+1}_\eta, d^{\alpha+1}_\eta)_{\eta \in T_{\alpha+1}}$, except s-indiscernibility and the coherence condition (4).

We let $(c^{\alpha+1}_\eta, d^{\alpha+1}_\eta)_{\eta \in T_{\alpha+1}}$ be an s-indiscernible tree over M, locally based on $(e_\eta, f_\eta)_{\eta \in T_{\alpha+1}}$.

By an automorphism, we may assume that

$$c^{\alpha+1}_{\iota_{\alpha\alpha+1}(\eta)} = c^{\alpha}_{\eta} \text{ for all } \eta \in T_\alpha .$$

This satisfies all of the requirements.

For δ limit, we define $(c^{\delta}_{\eta}, d^{\delta}_{\eta})_{\eta \in T_\delta}$ by setting

$$c^{\delta}_{\iota_{\alpha\delta}(\eta)} = c^{\alpha}_{\eta} \text{ for all } \alpha < \delta,\ \eta \in T_\alpha$$

which is well-defined by (4) and is easily seen to satisfy the requirements.

Continuing our construction until we arrive to $(c^{\kappa}_{\eta}, d^{\kappa}_{\eta})_{\eta \in T_\kappa}$ for κ sufficiently large, we apply Erdős-Rado to conclude. $\blacksquare$

The Independence Theorem

Theorem

Suppose T is $NSOP_1$. If $M \models T$, $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_M c$, and $b \mathop{\smile\!\!\!\!\!\!\mid}^K_M c$, then there is a_* such that $a_* \equiv_{Mb} a$, $a_* \equiv_{Mc} a'$, and $a_* \mathop{\smile\!\!\!\!\!\!\mid}^K_M bc$.

Proof

Let $p_0(x;y) = tp(a;b/M)$

$p_1(x;z) = tp(a';c/M)$.

We want to show

$$p_0(x;b) \cup p_1(x;c)$$

does not Kim-fork over M.

Towards contradiction, suppose

$$p_0(x;b) \cup p_1(x;c)$$

Kim-forks over M. Let $q \supseteq tp(b/M)$ be a global M-finitely satisfiable type. By the zig-zag lemma, there is a tree Morley sequence $(b_i, c_i)_{i \in \mathbb{Z}}$ such that

- $i \leq j \Rightarrow b_i c_j \equiv_M bc$
- $i > j \Rightarrow b_i \models q|_{Mc_j}$.

We know $\langle (b_{2i}, c_{2i}, b_{2i+1}, c_{2i+1}) : i \in \mathbb{Z} \rangle$ is also a tree Morley sequence over M. Because passing to a subtuple preserves tree Morley-ness, it follows that

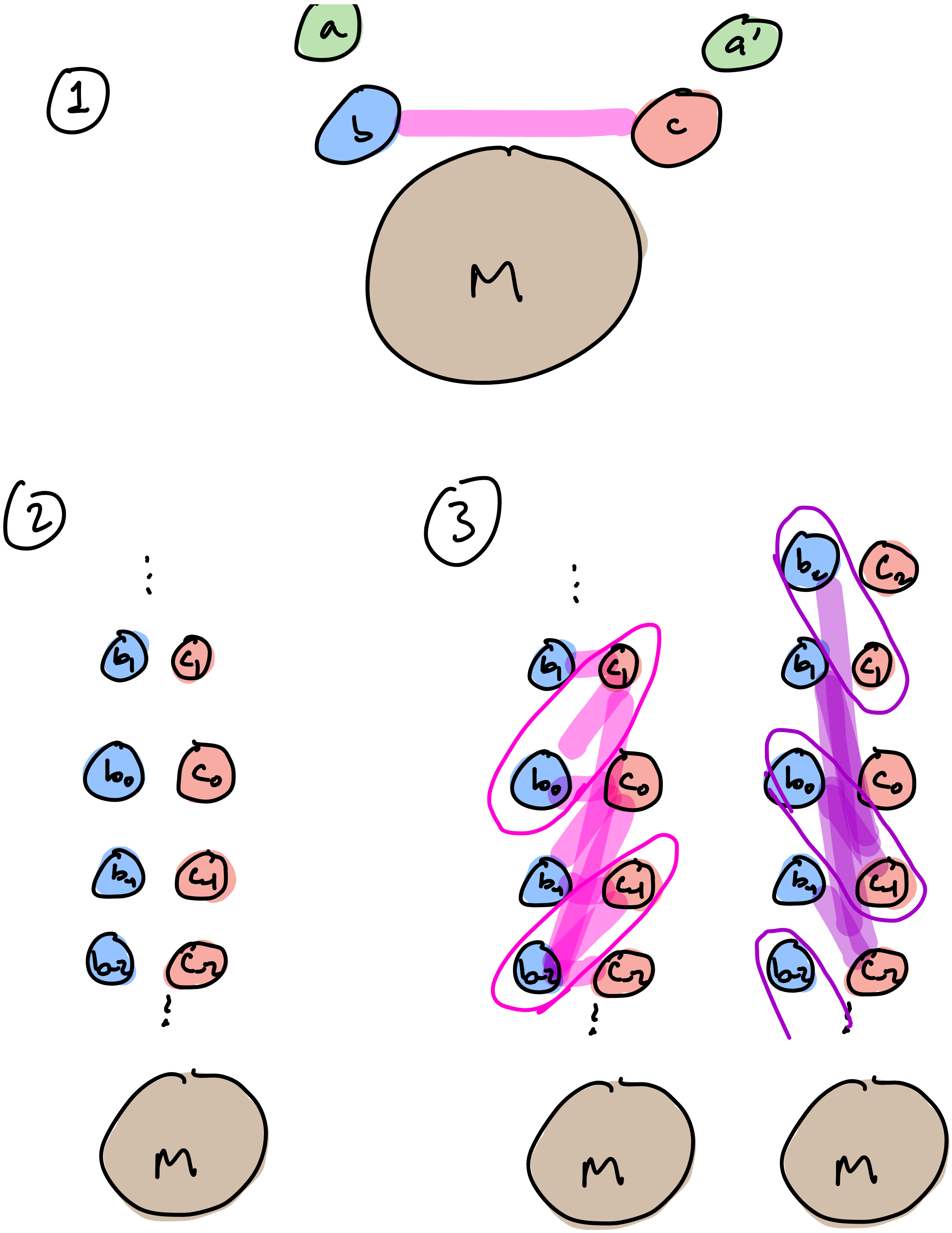
$$\langle (b_{2i}, c_{2i+1}) : i \in \mathbb{Z} \rangle.$$

is a tree Morley sequence over M.

Arguing similarly,

$$\langle (b_{2i}, c_{2i-1}) : i \in \mathbb{Z} \rangle$$

is a tree Morley sequence over M.



1
a
a'
b
c
M
2
b1
c1
b0
c0
b-1
c-1
b-2
c-2
M
3
b2
c2
M
M

By the choice of the sequence, $b_0 c_1 \equiv_M bc$, so we know $p_0(x;b_0) \cup p_1(x;c_1)$ Kim-forks over M. By Kim's Lemma for tree Morley sequences,

$$\bigcup_{i \in \mathbb{Z}} p_0(x;b_{2i}) \cup p_1(x;c_{2i+1})$$

is inconsistent.

On the other hand, $b_0 \models q|_{Mc_{-1}}$ hence $b_0 \mathop{\smile\!\!\!\!\!\!\,|}^u_M c_{-1}$. It follows from the weak independence theorem, $p_0(x;b_0) \cup p_1(x;c_{-1})$

is not Kim-forking over M,

so

$$\bigcup_{i \in \mathbb{Z}} p_0(x; b_{2i}) \cup p_1(x; c_{2i-1})$$

is consistent. But

$$\bigcup_{i \in \mathbb{Z}} p_0(x; b_{2i}) \cup p_1(x; c_{2i+1})$$

$$= \bigcup_{i \in \mathbb{Z}} p_0(x; b_{2i}) \cup p_1(x; c_{2i-1}).$$

Contradiction. $\square$

Finding Tree Morley Sequences

Proposition

Suppose T is $NSOP_1$, $M \models T$. If $b \equiv_M b'$ and $b \mathop{\smile\!\!\!\!\!\!|}^K_M b'$ then there is a tree Morley sequence over M, $(b_i)_{i<\omega}$, such that $b_0 = b'$ and $b_1 = b$.

Proof By induction on $\alpha \geq 1$, we will construct $(b^\alpha_\eta)_{\eta \in T_\alpha}$ such that

(1) $(b^\alpha_\eta)_{\eta \in T_\alpha}$ is s-indiscernible and spread out over M.

(2) If $\nu \triangleleft \eta$ then $b^\alpha_\nu b^\alpha_\eta \equiv_M b b'$.

(3) If $\beta < \alpha$, then

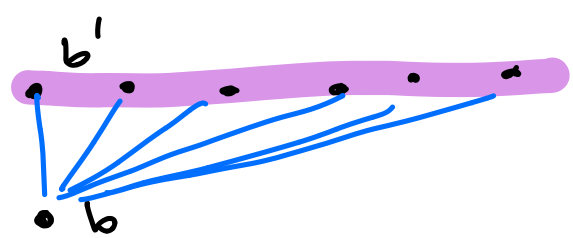
$$b^{\alpha}_{\iota_{\beta\alpha}(\eta)} = b^{\beta}_{\eta} \text{ for all } \eta \in T_{\beta}.$$

To begin, let $I = \langle b'_i \mid i < \omega \rangle$ be a Morley sequence over M in some global M-invariant type, with $b'_0 = b'$. As $b \mathop{\smile\!\!\!\!\!\!\,|}^{K}_{M} b'$, we may assume I is Mb-indiscernible.

Then we may define $(b_\eta)_{\eta \in T_1}$ by

$$b_\emptyset = b$$

$$b_{\langle i \rangle} = b'_i \text{ for all } i < \omega.$$



Given $(b^\beta_\eta)_{\eta \in T_\beta}$ for all $\beta \leq \alpha$, we will let $\langle (b^\alpha_{\eta,i})_{\eta \in T_\alpha} \mid i < \omega \rangle$ be a Morley sequence over M in some global M-invariant type, with

$$(b^\alpha_\eta)_{\eta \in T_\alpha} = (b^\alpha_{\eta,0})_{\eta \in T_\alpha}.$$

Let $p(x,y) = tp(b, b'/M)$. By our "consistency along trees" Lemma, we know $\bigcup_{\eta \in T_\alpha} p(x; b^\alpha_\eta)$ is non-Kim-forking over M. Hence

$$\bigcup_{i < \omega} \bigcup_{\eta \in T_\alpha} p(x; b^\alpha_{\eta,i})$$

is consistent. Let b_* be a

realization. Define a tree $(c_\eta)_{\eta \in T_{\alpha+1}}$ by $c_\emptyset = b_*$ and $c_{\langle i \rangle ^\frown \eta} = b^\alpha_{\eta, i}$ for all $i < \omega$ and $\eta \in T_\alpha$. We will let $(b^{\alpha+1}_\eta)_{\eta \in T_{\alpha+1}}$ be s-indiscernible over M locally based on $(c_\eta)_{\eta \in T_{\alpha+1}}$. By an automorphism, we may assume

$$b^{\alpha+1}_{\iota_{\alpha\alpha+1}(\eta)} = b^\alpha_\eta \text{ for all } \eta \in T_\alpha.$$

This satisfies the constraints.

For limit δ, we define

$$b^\delta_{\iota_{\alpha\delta}(\eta)} = b^\alpha_\eta \quad \text{for all } \alpha < \delta \text{ and } \eta \in T_\alpha.$$

This is well-defined by (3) and satisfies the constraints. Applying Erdős-Rado to $(b^{\kappa}_{\eta})_{\eta \in T_\kappa}$ for a cardinal κ sufficiently large, we obtain a tree Morley sequence over M $(b_i)_{i<\omega}$ such that $b_0 b_1 \equiv_M b' b$. By an automorphism, we may assume $b_0 = b'$, $b_1 = b$. ∎

Strengthened Independence

Theorem

Note that in a simple theory, if $a_* \mathop{\smile\!\!\!\!\!\!|}_M bc$ and $b \mathop{\perp}_M c$,

then, by base monotonicity, we have $a_* \mathop{\smile\!\!\!\!\!\!|}_{Mb} c$ + $b \mathop{\perp}_M c$,

hence $a_* b \mathop{\perp}_M c$, and thus $c \mathop{\smile\!\!\!\!\!\!|}_M a_* b$ as well. A similar argument gives $b \mathop{\perp}_M a_* c$.

Can one arrange this in NSOP$_1$

theories?

Theorem (Kruckman–R.)

Suppose T is $NSOP_1$, $M \models T$, $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!\!\!\;|}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!\;|}^K_M c$, and $b \mathop{\smile\!\!\!\!\!\!\;|}^K_M c$. Then there is a_* such that $a_* \equiv_{Mb} a$, $a_* \equiv_{Mc} a'$, $a_* \mathop{\smile\!\!\!\!\!\!\;|}^K_M bc$, $b \mathop{\smile\!\!\!\!\!\!\;|}^K_M a_* c$, and $c \mathop{\smile\!\!\!\!\!\!\;|}^K_M a_* b$.

Proof

By extension, there is $b' \equiv_{Mc} b$ such that $b' \mathop{\smile\!\!\!\!\!\!\;|}^K_M bc$. Let $\sigma \in Aut(\mathbb{M}/Mc)$ be an automorphism with $\sigma(b) = b'$.

Then let $b'' = \sigma^{-1}(b)$. Then we have

$$b' \mathop{\perp\!\!\!\perp}^{K}_{M} bc \Longleftrightarrow \sigma(b') \mathop{\perp\!\!\!\perp}^{K}_{M} \sigma(b)c$$

$$\Longleftrightarrow b \mathop{\perp\!\!\!\perp}^{K}_{M} b''c.$$

By symmetry, we have $b''c \mathop{\perp\!\!\!\perp}^{K}_{M} b$.

Pick c' such that $b''c \equiv_M bc'$.

By extension, there is

$$b''_* c_* \equiv_{Mb} b''c$$

such that $b''_* c_* \mathop{\perp\!\!\!\perp}^{K}_{M} bc'$.

Let $\tau \in \mathrm{Aut}(\mathbb{M}/_{Mb})$ be an automorphism with $T(b''_* c_*) = b''c$.

Then we have

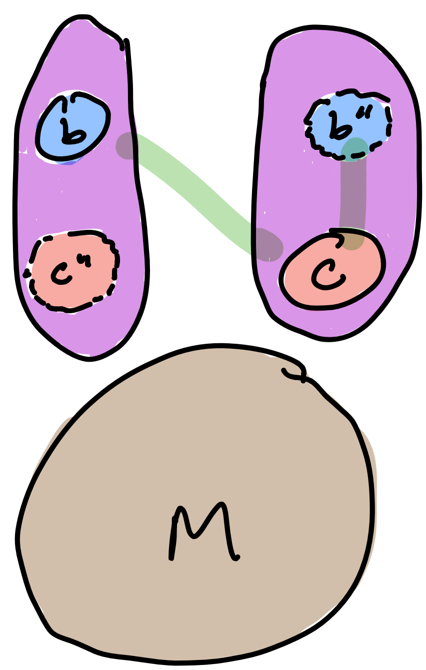
$$b''_* c_* \underset{M}{\overset{K}{\smile\!\!\!\!\!\!\mid}} bc' \iff T(b''_* c_*) \underset{M}{\overset{K}{\smile\!\!\!\!\!\!\mid}} bT(c')$$

$$\iff b''c \underset{M}{\smile\!\!\!\!\!\!\mid} bT(c').$$

Let $c'' = T(c')$. Now we have

$b''c \equiv_M bc''$, $b'' \equiv_{Mc} b$,

and $b''c \underset{M}{\overset{K}{\smile\!\!\!\!\!\!\mid}} bc''$!



It follows that there is a tree Morley sequence over M $\langle (b_i, c_i) : i \in \mathbb{Z} \rangle$ such that $b_0 = b$, $c_1 = c$.

By the independence theorem, there is a_0 such that $a_0 \equiv_{Mb} a$, $a_0 \equiv_{Mc} a'$, and $a_0 \mathop{\smile\!\!\!\!\!\!\;\;|}^{K}_{M} bc$.

Choose a_1 such that

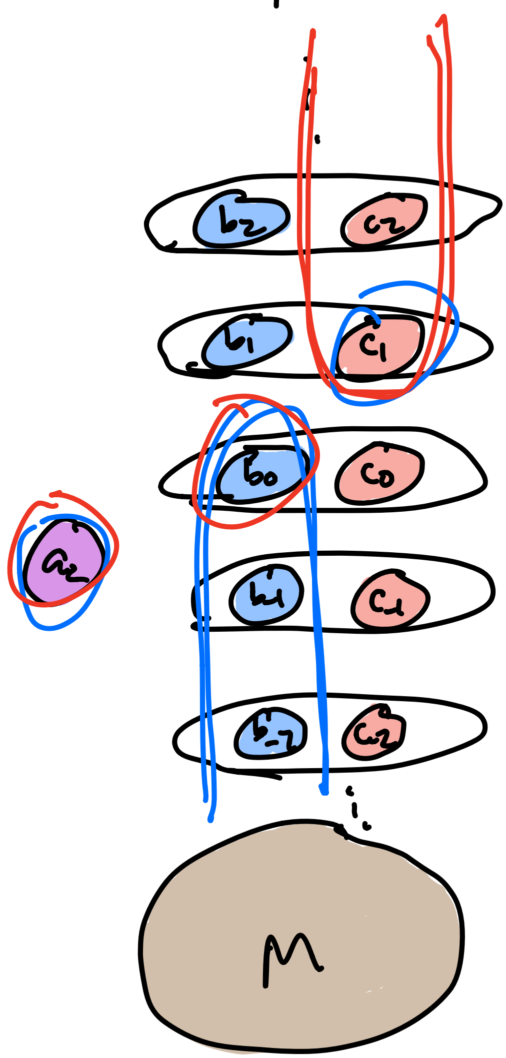
$$a_1 b_1 c_1 \equiv_M a_0 b c$$

(possible since $b_1 = b''$ and $b'' \equiv_{Mc} b$).

Then $a_1 \mathop{\smile\!\!\!\!\!\!\;\;|}^{K}_{M} b_1 c_1$ so, by the chain condition for tree Morley sequences over M, there is $a_2 \equiv_{Mb_1c_1} a_1$ such that I is Ma_2-indiscernible and $a_2 \mathop{\smile\!\!\!\!\!\!\mid}^K_M I$. In particular, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}^K_M bc$. However, we know $(b_i)_{i \leq 0}$ is a tree Morley sequence over M with $b_0 = b$ which is $Ma_2 I_{>0}$-indiscernible and therefore Ma_2c-indiscernible. This show $a_2 c \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$. Likewise

$(c_i)_{i \geq 1}$ is a tree Morley sequence over M with $c_1 = c$ which is Ma_2b-indiscernible, so $a_2 b \mathop{\smile\!\!\!\!\!\!\;|}^{K}_{M} c$.

We conclude by symmetry. ∎



Corollary of the Independence Theorem

Corollary Assume T is $NSOP_1$ and $M \models T$. Suppose $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$ and $I = (b_i)_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!\mid}^K$-Morley sequence over M — ie. $(b_i)_{i<\omega}$ is M-indiscernible and $b_i \mathop{\smile\!\!\!\!\!\!\mid}^K_M b_{<i}$ for all $i<\omega$. Then there is $a' \equiv_{Mb} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_M I$ and I is Ma'-indiscernible.

Proof Let $p(x,y) = tp(a,b/M)$.

First, we will prove by induction on $n<\omega$ that $\bigcup_{i\leq n} p(x;b_i)$ is non-Kim-forking over M. This is clear for $n=0$.

Then, assuming it for n, we can pick

$a_n \models \bigcup_{i \leq n} p(x; b_i)$ and $a' \models p(x; b_{n+1})$.

Since $b_{n+1} \mathop{\smile\!\!\!\!\!\!|}^K_M b_{\leq n}$, $a_n \mathop{\smile\!\!\!\!\!\!|}^K_M b_{\leq n}$,

$a' \mathop{\smile\!\!\!\!\!\!|}^K_M b_{n+1}$, and $a_n \equiv_M a'$, we may

apply the independence theorem to find

some $a_{n+1} \models \bigcup_{i \leq n+1} p(x; b_i)$, with

$a_{n+1} \mathop{\smile\!\!\!\!\!\!|}^K_M b_{\leq n+1}$.

By compactness, it follows that

$\bigcup_{i < \omega} p(x; b_i)$ is consistent and

non-Kim-forking over M. Let a_*

be a realization. Then by

Ramsey, compactness, and an automorphism, we may assume I is

Ma_*-indiscernible. $\blacksquare$