Forking and Dividing at a Generic Scale

An Introduction to Kim-Independence and $NSOP_1$

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Lecture 2

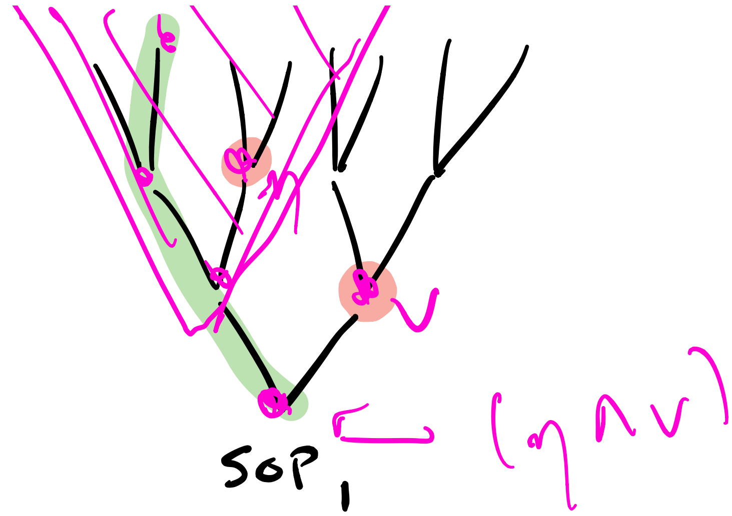
Last time

Definition

$\varphi(x;y)$ has SOP$_1$ if there is $(a_\eta)_{\eta \in 2^{<\omega}}$ such that

- (paths are consistent) For all $\eta \in 2^{\omega}$
  $$\{\varphi(x;a_{\eta|i}) : i<\omega\}$$
  is consistent.

- (Inconsistency) For all $\eta, \nu \in 2^{<\omega}$, if $\eta \trianglerighteq (\eta \wedge \nu)^\frown \langle 0 \rangle$, $\nu = (\eta \wedge \nu)^\frown \langle 1 \rangle$
  $$\{\varphi(x;a_\eta), \varphi(x;a_\nu)\}$$
  is inconsistent.



SOP_1

We say T has SOP_1 if some formula has SOP_1 modulo T; we say T is $NSOP_1$ if its not SOP_1

SOP_1 Arrays

Proposition

The following are equivalent:

(1) T has SOP_1

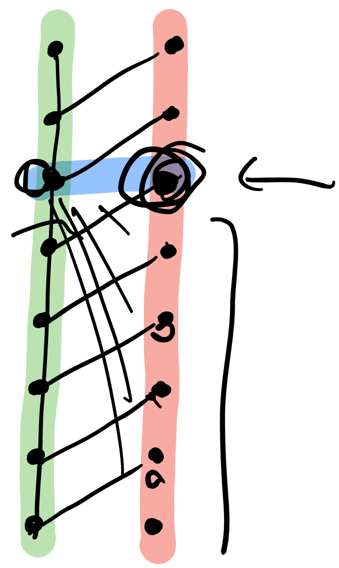
(2) There is $\varphi(x;y)$, $k<\omega$, and $(c_{i,0}, c_{i,1})_{i<\omega}$ such that

(a) $\{\varphi(x;c_{i,0}) : i<\omega\}$ is consistent.

(b) $\{\varphi(x;c_{i,1}) : i<\omega\}$ is k-inconsistent.

(c) For all $i<\omega$, $c_{i,0} \equiv_{c_{<i,0}\, c_{<i,1}} c_{i,1}$

(3) Same as (2), but with $k=2$.



SOP_1

Note The equivalence of (1) and (3) also holds for formulas. Also, since any φ will witness SOP_1 also in any expansion of our ambient monster $\mathbb{M}$ (in the language L, say), we may find such an array with $c_{i,0} \equiv^{L'}_{c_{\triangleleft i,0}\, c_{\triangleleft i,1}} c_{i,1}$ where L' is a larger language, once $\mathbb{M}$ has been expanded to an L'-structure $\mathbb{M}'$.

Invariant Types

monster model of T

Recall $p(x) \in S_x(\mathbb{M})$ is called **A-invariant** if $\sigma p = p$ for all $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ or, equivalently, for all $\varphi(x;y)$, if $c \equiv_A c'$, then

$$\varphi(x;c) \in p \iff \varphi(x;c') \in p$$

Given A-invariant $p \in S_x(\mathbb{M})$, $q \in S_y(\mathbb{M})$, we obtain an A-invariant type $(p \otimes q)(x,y) \in S_{x,y}(\mathbb{M})$ defined by

$$(p \otimes q) = tp(a,b/\mathbb{M})$$

for some $b \models q|_M$ and $a \models p|_{Mb}$.

Given any linear order I and A-invariant $p \in S_x(M)$, we have another A-invariant global type $p^{\otimes I}(x_\alpha : \alpha \in I)$ such that

$(a_\alpha)_{\alpha \in I} \models p^{\otimes I} \iff a_\alpha \models p|_{M a_{<\alpha}}$ for all $\alpha \in I$.

If $B \supseteq A$ and $(a_\alpha)_{\alpha \in I} \models p^{\otimes I}|_B$, then $(a_\alpha)_{\alpha \in I}$ is B-indiscernible.

When I is an infinite linear order, we refer to $(a_\alpha)_{\alpha \in I}$ as a Morley sequence in p (over B).

Note: Being a Morley sequence over A in a global A-invariant type p implies being a Morley sequence over A in the sense of the previous lecture, i.e.

$$(a_\alpha)_{\alpha \in I} \models p^{\otimes I}\big|_A \Rightarrow a_\alpha \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{f}_{A} a_{<\alpha} \text{ for all } \alpha \in I.$$

But the converse is usually false.

Average Types

If A is a set of parameters and $\mathcal{D}$ is an ultrafilter on A^n, then for any B,

$$Av(D,B) = \{\varphi(x) \in L(B) \mid \{a \in A^n \mid \models \varphi(a)\} \in D\}$$

is called the average type of D over B.

(1) $Av(D,B)$ is a complete type over B.

(2) $Av(D,\mathbb{M})$ is a global A-invariant type (in fact, A-finitely satisfiable).

(3) If $M \models T$ and $p \in S_x(M)$, there is some D on $M^{|x|}$ such that $Av(D,M) = p$.

Together, these imply the following

Fact

"Types over models have global invariant extensions"

If $M \models T$ and $p \in S_x(M)$, then there is a global M-invariant (even M-finitely satisfiable) $q \in S_x(\mathbb{M})$ with $q|_M = p$.

Proof Pick an ultrafilter D on $M^{|x|}$ such that $Av(D, M) = p$.

Then $Av(D, \mathbb{M})$ is the desired global extension.

Note

This fact implies that, over models, in every type, we have some notion of a 'generic sequence'— namely, a 'generic sequence in p', for some $p \in S_x(M)$, should be a Morley sequence over M in some global M-invariant type extending p.

Kim-Independence

Definition

Fix $M \models T$.

(1) We say $\varphi(x;a)$ **Kim-divides over M** if there is a global M-invariant $p \supseteq tp(a/M)$ such that φ divides with respect to a Morley sequence over M in p (or, equivalently, in every Morley sequence over M in p).

(2) We say $\varphi(x;a)$ Kim-forks over M if $\varphi(x;a) \vdash \bigvee_{i<k} \psi_i(x;c_i)$, where $\psi_i(x;c_i)$ Kim-divides over M.

(3) We say a type $p(x)$ Kim-divides or Kim-forks over M if p implies some formula that Kim-divides or Kim-forks over M, respectively.

(4) We write $a \mathop{\smile\!\!\!\!\!\!\mid}^{K}_{M} b$ to indicate $tp(a/Mb)$ does not Kim-fork over M.

Kim's lemma

Theorem (Kaplan–R.)

The following are equivalent:

(1) T is $NSOP_1$

(2) Kim's lemma for Kim-dividing

If $M \models T$ and $\varphi(x;a)$ Kim-divides over M, then $\varphi(x;a)$ divides with respect to Morley sequences over M in every global M-invariant type $p \supseteq tp(a/M)$.

(3) If $M \models T$, then if $\varphi(x;a)$ divides with respect to a Morley sequence over M in some global M-finitely satisfiable type $p \supseteq tp(a/M)$, then $\varphi(x;a)$ divides with respect to Morley sequences over M in every global M-finitely satisfiable $q \supseteq tp(a/M)$.

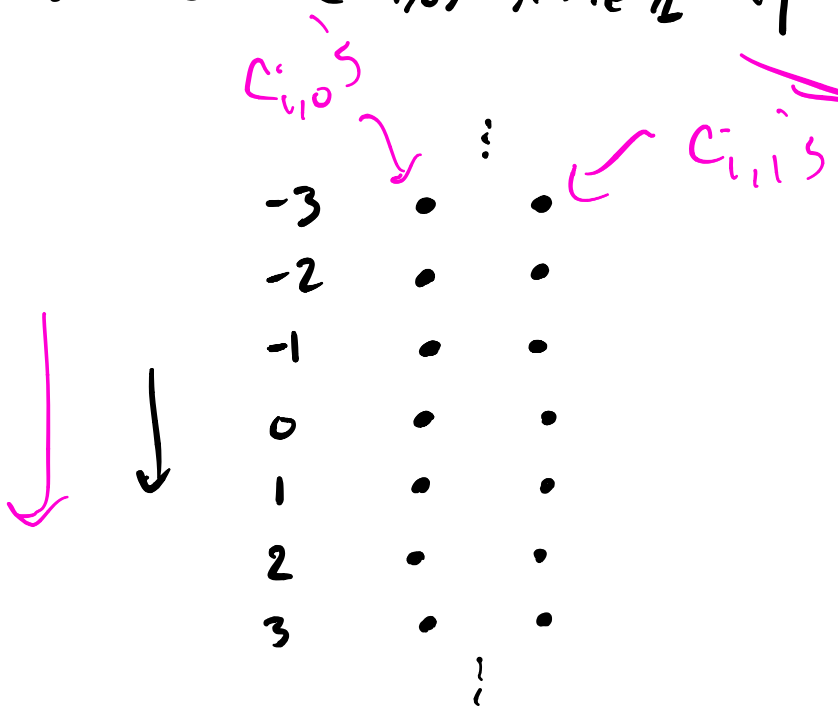
Proof

$(2)\Rightarrow(3)$ follows from the fact that M-finitely satisfiable types are M-invariant.

$(1)\Rightarrow(2)$ Assume (2) fails and we will show T has SOP_1. So fix $M \models T$, $\varphi(x;a)$, and global M-invariant $p, q \supseteq tp(a/M)$ such that $\varphi(x;a)$ divides with respect to Morley sequences over M in q, but $\varphi(x;a)$ does not divide with respect to Morley sequences over

M in p.

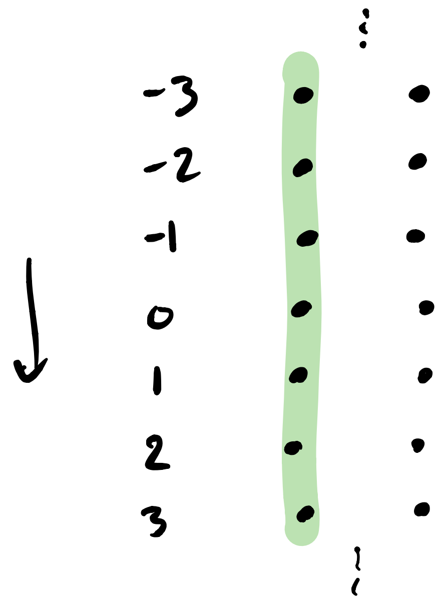
Now let $(c_{i,0}, c_{i,1})_{i\in\mathbb{Z}} \models (p\otimes q)^{\otimes\mathbb{Z}}|_M$.



The rest of the proof will be a reflection on this picture.

Observation 1: $(c_{i,0})_{i\in\mathbb{Z}} \models p^{\otimes\mathbb{Z}}|_M$

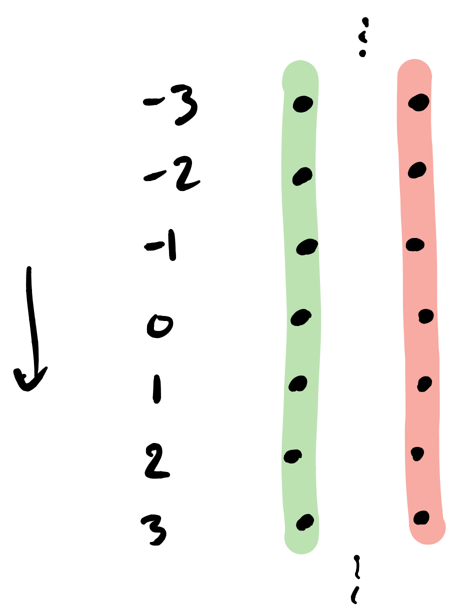
so $\{\varphi(x; c_{i,0}) : i\in\mathbb{Z}\}$ is consistent.



Observation 2: $(c_{1,i})_{i\in\mathbb{Z}} \models q^{\otimes\mathbb{Z}}|_M$

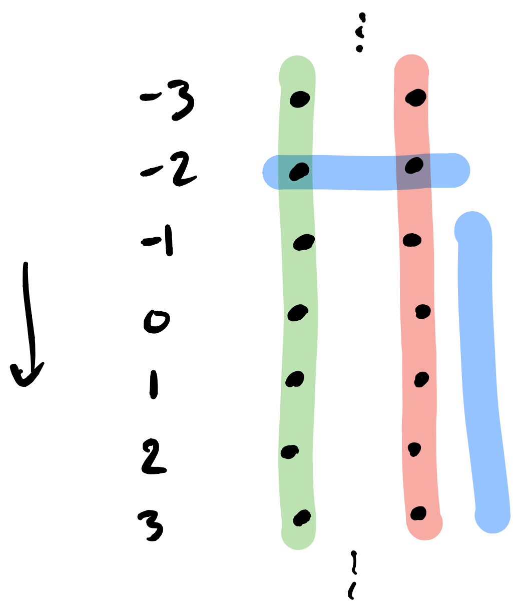
so $\{\varphi(x; c_{1,i}) : i \in \mathbb{Z}\}$ is k-inconsistent

for some k.



Observation 3: For all $j \in \mathbb{Z}$,

$$c_{j,0} \equiv_{M c_{>j,0} c_{>j,1}} c_{j,1} .$$



Why? Notice

$$(c_{i,0}, c_{i,1})_{i \in \mathbb{Z}} \models (p \otimes q)^{\otimes \mathbb{Z}} / M$$

So

$$(c_{i,0}, c_{i,1})_{i > j} \models (p \otimes q)^{\otimes \omega} / \underbrace{M c_{j,0}, c_{j,1} \ldots}$$

Moreover, $c_{j,0} \models p|_M$, $c_{j,1} \models q|_M$ and

$p|_M = q|_M = tp(a/M)$ so

$$c_{j,0} \equiv_M c_{j,1}.$$

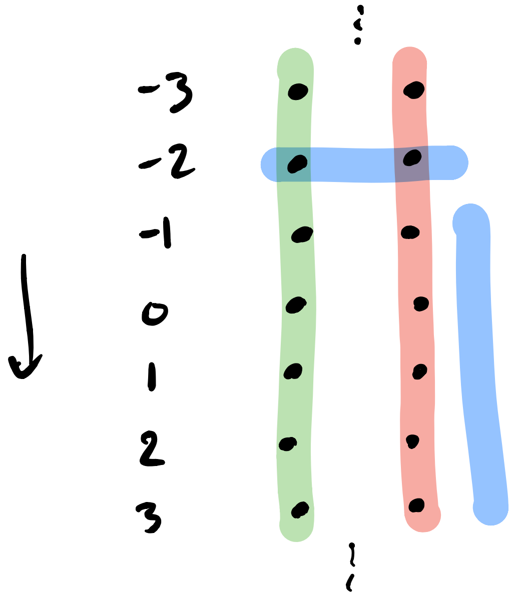
Because $(p \otimes q)^{\otimes \omega}$ is M-invariant,

we know, for all $\varphi(\bar{z}; y) \in L(M)$

$$\varphi(\bar{z}; c_{j,0}) \in (p \otimes q)^{\otimes \omega} \Leftrightarrow \varphi(\bar{z}; c_{j,1}) \in (p \otimes q)^{\otimes \omega}$$

so we have

$$c_{j,0} \equiv_{M c_{>j,0} c_{>j,1}} c_{j,1}.$$



So we've found an SOP_1 array, upside down. [To put it right side up, define $(d_{i,0}, d_{i,1})_{i<\omega}$ by

$$(d_{i,0}, d_{i,1}) = (c_{-i,0}, c_{-i,1})$$

for all $i<\omega$.] This shows

$$\neg(2) \Rightarrow \neg(1) \quad \text{so} \quad (1) \Rightarrow (2).$$

(1) $\Rightarrow$ (3) Assume T has SOP_1. Let T^{sk} be an expansion of T to a larger language that contains Skolem functions [— i.e. such that for every $\varphi(x;y)$ in L^{sk} with $|x|=1$ there is a function $f_\varphi(y)$ such that

$$T^{sk} \vdash (\forall y)\Big[(\exists x)\varphi(x;y) \rightarrow \varphi(f_\varphi(y);y)\Big].$$]

Then there is an L-formula $\varphi(x;y)$ and a collection of tuples $(c_{i,0}, c_{i,1})_{i<\omega}$ such that

- $\{\varphi(x; c_{i,0}) : i < \omega\}$ is consistent.
- $\{\varphi(x; c_{i,1}) : i < \omega\}$ is 2-inconsistent.
- $c_{i,0} \equiv^{L^{Sk}}_{c_{<i,0}\, c_{<i,1}} c_{i,1}$ for all $i < \omega$.

By Ramsey and compactness, we may assume further that $\langle (c_{i,0}, c_{i,1}) : i < \omega \rangle$ is an L^{Sk}-indiscernible sequence.

Let $M = \mathrm{dcl}_{L^{Sk}}((c_{i,0}, c_{i,1})_{i<\omega})$.

Then $M \restriction L \models T$.

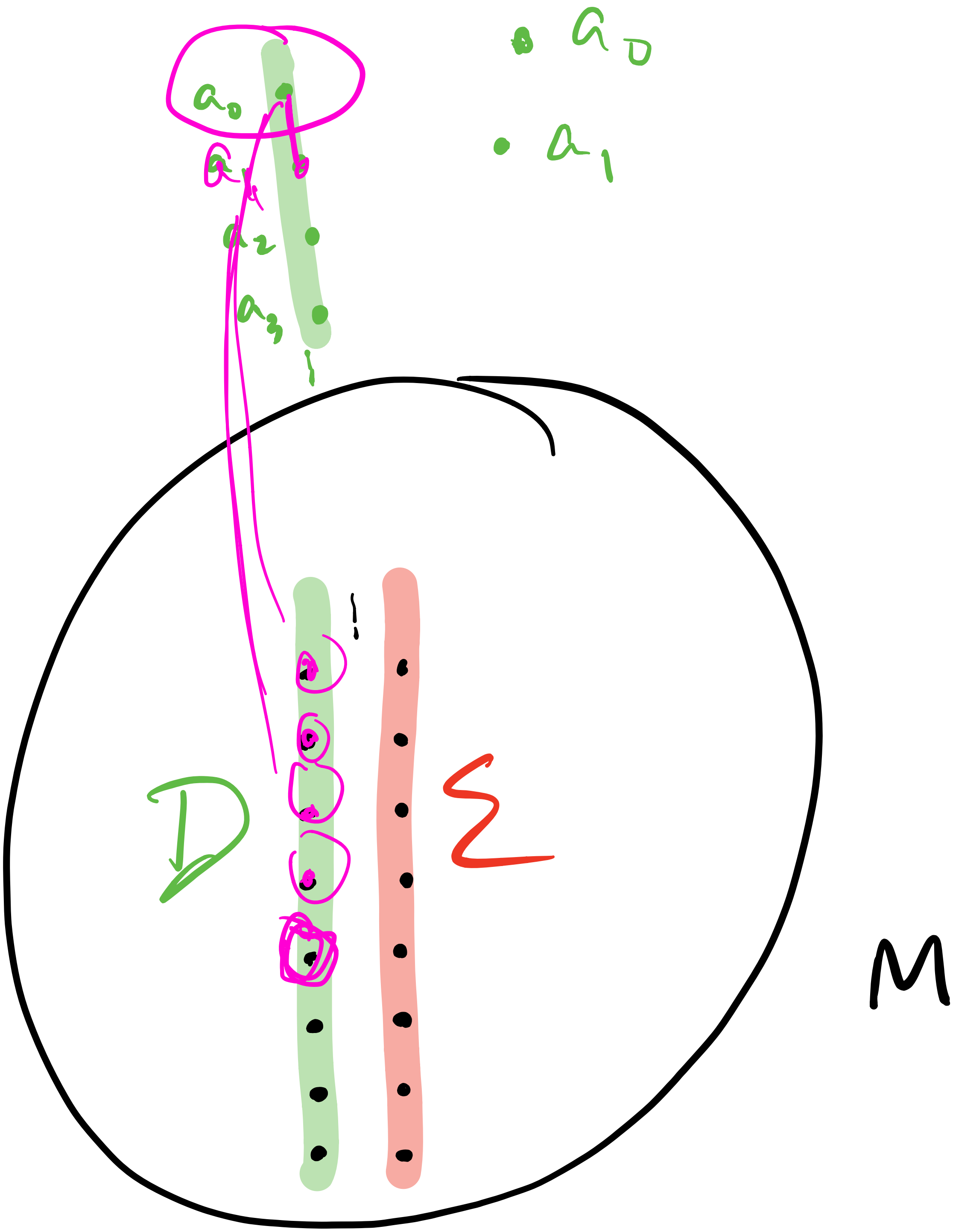
Let $n = |c_{i,0}| = |c_{i,1}|$, and let $\mathcal{D}$ be a non-principal ultrafilter on M^n, concentrating on $\{c_{i,0} : i < \omega\} \subset M^n$

and let $\mathcal{E}$ be a non-principal ultrafilter concentrating on $\{c_{i,1} : i<\omega\}$.

Let $p = Av(\mathcal{D}, M)$

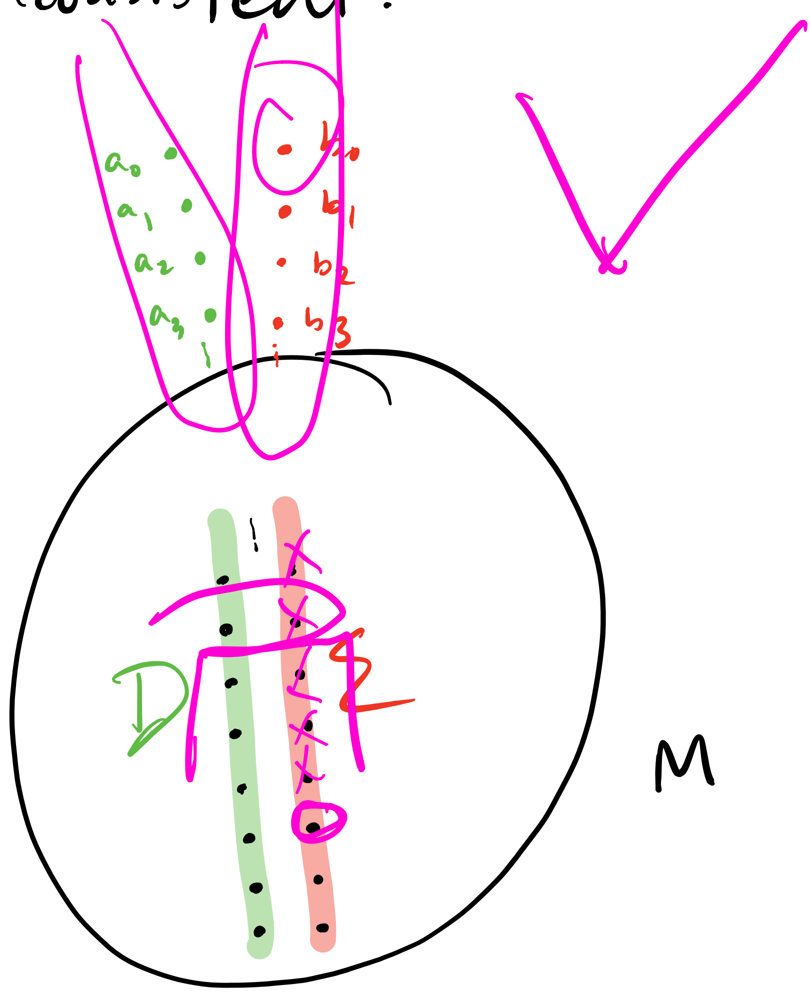
$q = Av(\mathcal{E}, M)$.

Observation 1: If $(a_i)_{i<\omega}$ is a Morley sequence over M in p then $\{\varphi(x;a_i) : i<\omega\}$ is consistent.



a_0
a_1
a_2
a_3
a_D
D
Σ
M

Observation 2: If $(b_i)_{i<\omega}$ is a Morley sequence over M in q, then $\{\varphi(x; b_i) : i<\omega\}$ is 2-inconsistent.



Observation 3:

$$p|_M = q|_M.$$

Suppose $\varphi(x;m) \in \underline{L(M)}$. Then there is a tuple of L^{sk}-terms t and some k such that $\varphi(x;m)$ is equivalent modulo T^{sk} to

$$\varphi(x;t(c_{<k,0}\,c_{<k,1})).$$

Because $\underline{c_{i,0} \equiv^{L^{sk}}_{c_{<i,0}\,c_{<i,1}} c_{i,1}}$ for all $i<\omega$, we know, in particular

$$\underline{c_{j,0} \equiv^{L^{sk}}_{c_{<k,0}\,c_{<k,1}} c_{j,1}}$$

for all $j > k$. Hence

$$\psi(x; m) \in p|_M \Leftrightarrow \{c_{j,0} \mid \models \psi(c_{j,0}; m)\} \in D$$

$$\Longleftrightarrow \{c_{j,0} \mid \models \psi(c_{j,0}; t(c_{<k,0}, c_{<k,1}))\} \in D$$

$$\Leftrightarrow \{c_{j,0} \mid \models \psi(c_{j,0}, t(c_{<k,0}, c_{<k,1}))\} \supseteq \{c_{j,0} \mid j > k\}$$

$$\Leftrightarrow \{c_{j,1} \mid \models \psi(c_{j,1}; t(c_{<k,0}, c_{<k,1}))\} \supseteq \{c_{j,1} \mid j > k\}$$

$$\Leftrightarrow \{c_{j,1} \mid \models \psi(c_{j,1}; m)\} \in E$$

$$\Leftrightarrow \psi(x; m) \in q|_M. \qquad \blacksquare$$

Kim-forking = Kim-dividing

Proposition

Assume T is $NSOP_1$. Then if $M \models T$ and $\varphi(x;a)$ Kim-forks over M, then $\varphi(x;a)$ Kim-divides over M.

Proof Fix $M \models T$.

Suppose $\varphi(x;a) \vdash \bigvee_{i<k} \psi_i(x;c_i)$ where $\psi_i(x;c_i)$ Kim-divides over M for all $i<k$.

Let $q \supseteq tp(a, c_0, \ldots, c_{k-1}/M)$

be a global M-invariant type, and let

$$(a_i, c_{0,i}, c_{1,i}, \dots, c_{k-1,i}) \models q^{\otimes \omega}|_M.$$

Then $(a_i)_{i<\omega}$ and each $(c_{j,i})_{i<\omega}$ are also Morley sequences over M in global M-invariant types. By Kim's lemma for Kim-dividing, we know

$$\{\psi_j(x; c_{j,i}) : i<\omega\}$$

is inconsistent for all $j<k$.

We want to show

$$\{\varphi(x;a_i) : i<\omega\}$$

is inconsistent.

Towards contradiction, suppose not and let b be a realization.

Then we know, for all $i<\omega$,

$$\varphi(x;a_i) \vdash \bigvee_{j<k} \psi_j(x;c_{j,i})$$

so for all $i<\omega$, there is some $j(i)<k$ such that

$$\models \psi_{j(i)}(b;c_{j(i),i}).$$

By the pigeonhole principle, there is some $j_* < k$ such that the set $X := \{ i < \omega \mid j(i) = j_* \}$ is infinite. Then we have both $\{ \neg \psi_{j_*}(x; c_{j_*, i}) : i \in X \}$ is inconsistent (by indiscernibility) and that this set of formulas is realized by b, a contradiction. $\blacksquare$

The Kim-Pillay Criterion

$NSOP_1$

Theorem (Chernikov-R.)

Suppose $\mathop{\smile\!\!\!\!\!|}^{*}$ is an $\mathrm{Aut}(\mathbb{M})$-invariant ternary relation on small subsets of $\mathbb{M}$ satisfying, for all $M \models T$:

(1) Existence: $a \mathop{\smile\!\!\!\!\!|}^{*}_{M} M$ always holds

(2) Symmetry: $a \mathop{\smile\!\!\!\!\!|}^{*}_{M} b \iff b \mathop{\smile\!\!\!\!\!|}^{*}_{M} a$

(3) Monotonicity: $aa' \mathop{\smile\!\!\!\!\!|}^{*}_{M} bb' \Rightarrow a \mathop{\smile\!\!\!\!\!|}^{*}_{M} b$.

(4) Strong finite character: If $a \not\mathop{\smile\!\!\!\!\!|}^{*}_{M} b$, there is $\varphi(x;b) \in tp(a/Mb)$ such that

$a' \not\mathop{\perp}^{*}_{M} b$ for all a' with $\models \varphi(a'; b)$.

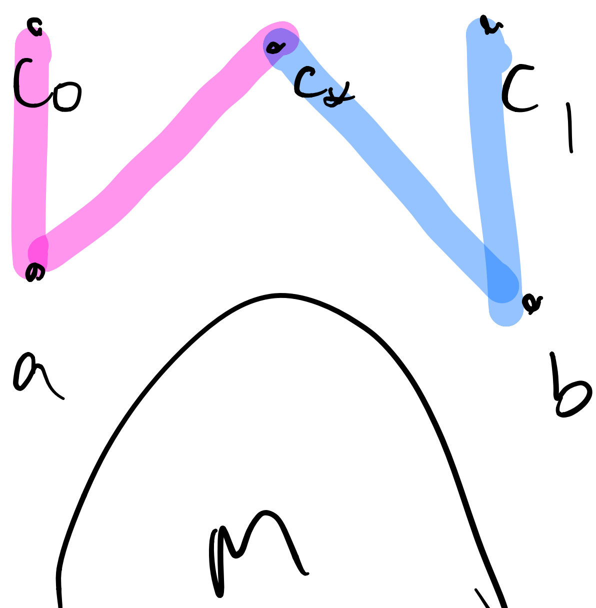
(5) The independence theorem.

If $c_0 \equiv_M c_1$, $c_0 \mathop{\perp}^{*}_{M} a$, $c_1 \mathop{\perp}^{*}_{M} b$, $a \mathop{\perp}^{*}_{M} b$,

then there is c_* such that

$c_* \equiv_{Ma} c_0$, $c_* \equiv_{Mb} c_1$, and $c_* \mathop{\perp}^{*}_{M} ab$.

Then, T is $NSOP_1$.



The Kim-Pillay Criterion

$NSOP_1$

Definition Suppose $M \models T$.

$a \mathop{\smile\!\!\!\!\!\!\|}^{u}_{M} b$ means $tp(a/Mb)$ extends to a global M-finitely satisfiable type. [Equiv to $tp(a/Ab)$ is M-fin sat].

$a \mathop{\smile\!\!\!\!\!\!\|}^{i}_{M} b$ means $tp(a/Mb)$ extends to a global M-invariant type.

Easy exercise If $M \models T$,

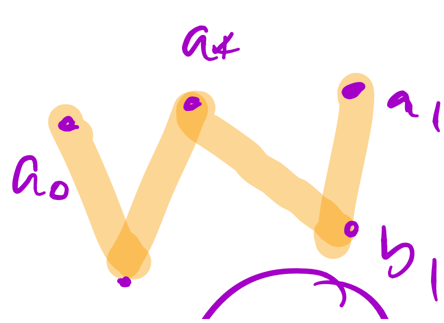
$$a \mathop{\perp}\limits^{u}_{M} b \Rightarrow a \mathop{\perp}\limits^{i}_{M} b \Rightarrow a \mathop{\perp}\limits^{f}_{M} b \Rightarrow a \mathop{\perp}\limits^{K}_{M} b.$$

Definition

Define $a \mathop{\perp}\limits^{h}_{M} b \Leftrightarrow b \mathop{\perp}\limits^{u}_{M} a$

and

$$a \mathop{\perp}\limits^{ci}_{M} b \Leftrightarrow b \mathop{\perp}\limits^{i}_{M} a.$$



Proposition

The following are equivalent:

(1) T is $NSOP_1$

(2) (Very weak IT for $\perp^h$)
If $M \models T$, $a_0 b_0 \equiv_M a_1 b_1$ and $a_i \perp^h_M b_i$, $i=0,1$, $b_0 \perp^h_M b_1$, then there is a_* with $a_* b_0 \equiv_M a_* b_1 \equiv_M a_0 b_0$.

(3) (Very weak IT for $\perp^{ci}$)
Same as (2) but with $\perp^{ci}$.

(3) $\Rightarrow$ (2) because $\perp^h \Rightarrow \perp^i$ hence $\perp^h \Rightarrow \perp^{ci}$.

(1) $\Rightarrow$ (3): Suppose (3) fails. So for some $M \models T$, there are $a_0 b_0 \equiv_M a_1 b_1$

with $a_i \underset{M}{\overset{ci}{\perp}} b_i$ for $i=0,1$, and

$b_0 \underset{M}{\overset{ci}{\perp}} b_1$, but, setting

$$p(x;y) = tp(a_0, b_0/M),$$

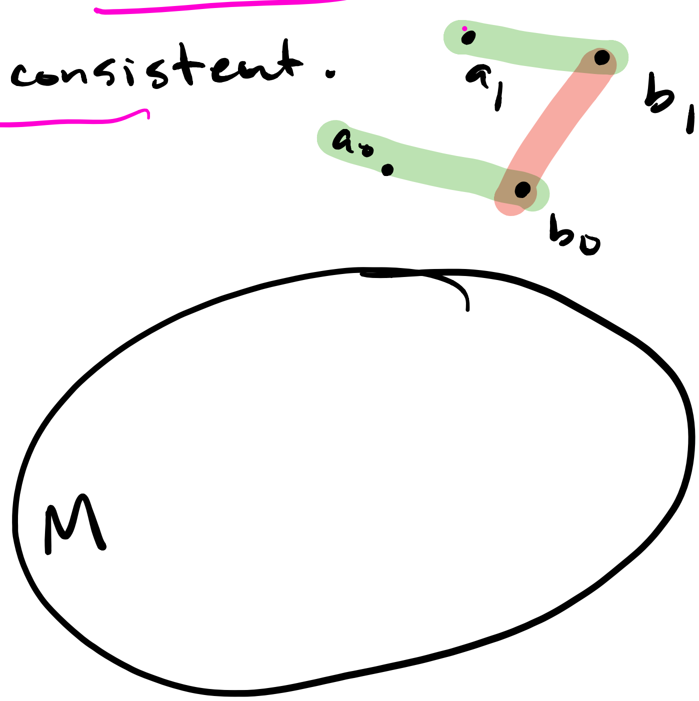
we have $p(x;b_0) \cup p(x;b_1)$ is

inconsistent. Let $\varphi(x;y) \in p$ be

a formula such that

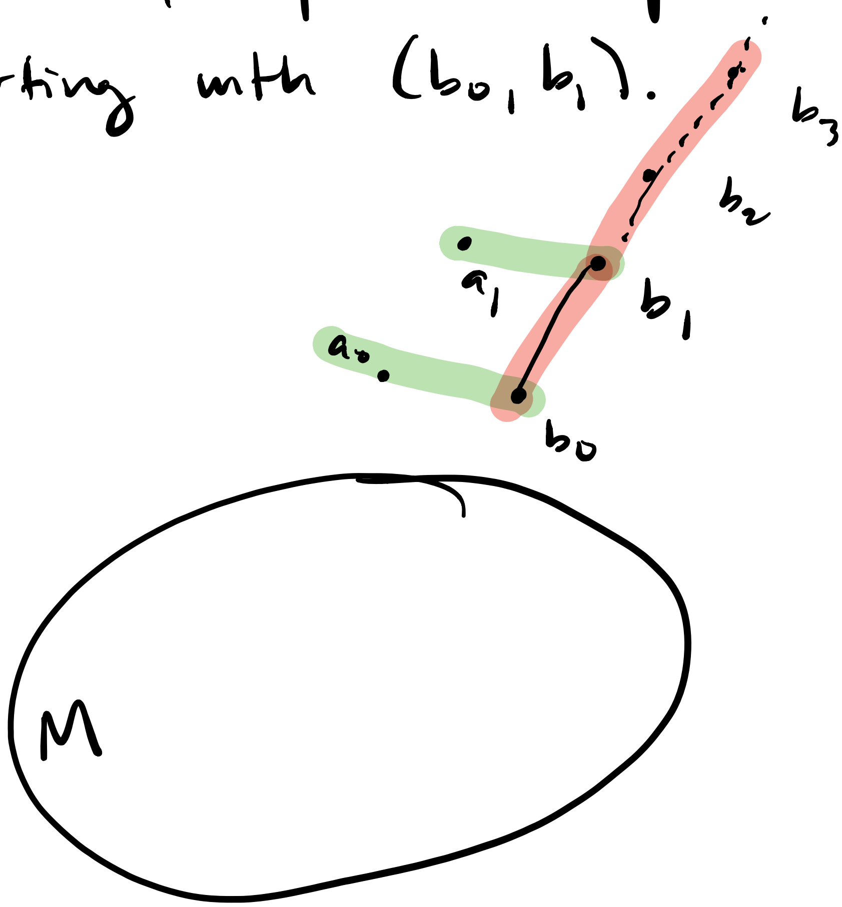
$$\{\varphi(x;b_0), \varphi(x;b_1)\}$$

is inconsistent.



Because $b_0 \underset{M}{\overset{ci}{\perp}} b_1$, we know $b_1 \underset{M}{\overset{i}{\perp}} b_0$ so there is some global M-invariant $q \supseteq tp(b_1/Mb_0)$.

Note that $b_0 \models q|_M$ so there is a Morley sequence in q over M starting with (b_0, b_1).



[Why?: Just choose $b_2 \models q|_{Mb_0b_1}$

$b_3 \models q|_{Mb_0b_1b_2}$

$\vdots$

and so on].

Then $\{\varphi(x;b_i) : i<\omega\}$ is 2-inconsistent

so $\varphi(x;b_0)$ Kim-divides over M.

But we also know $a_0 \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K}_{M} b_0$,

So there is a global M-invariant

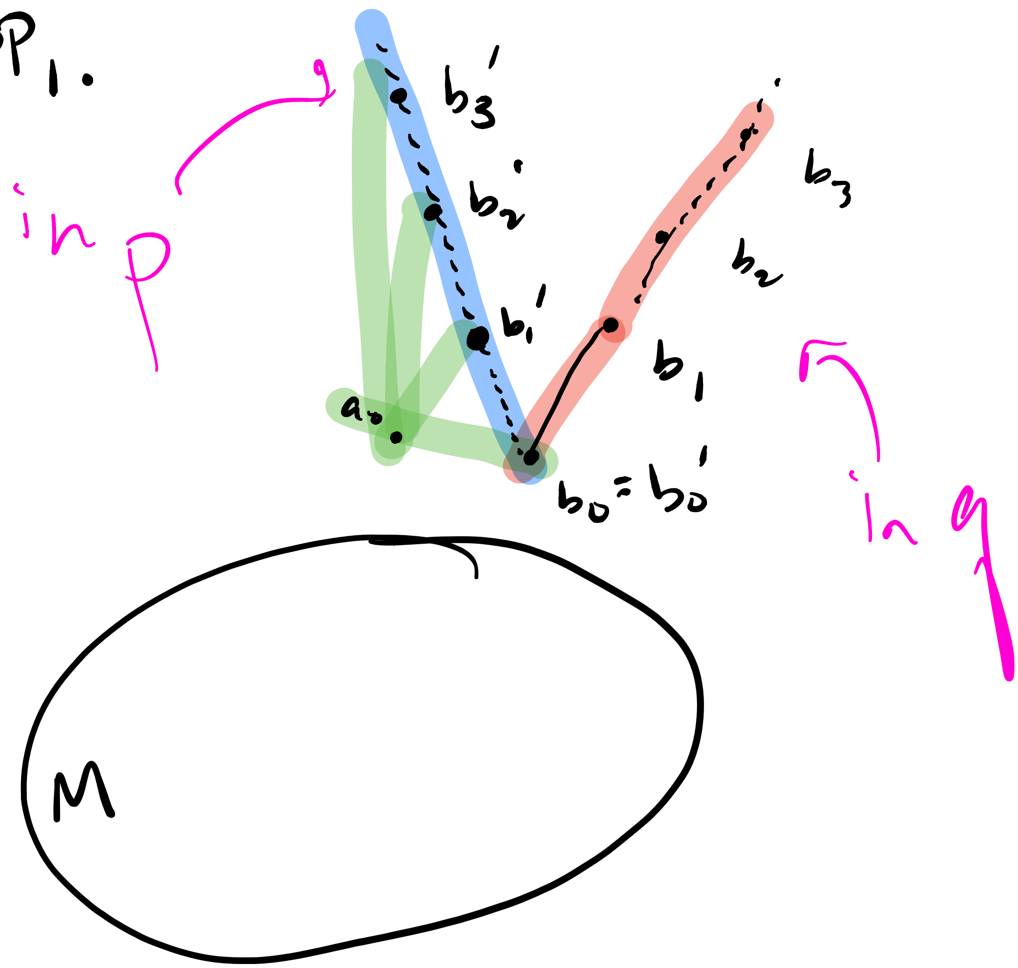
type $p \supseteq tp(b_0/Ma_0)$. Let

$(b_i')_{i<\omega} \models p^{\otimes\omega}|_{Ma}$ with $b_0' = b_0$.

Then $(b_i')_{i<\omega}$ is a Morley sequence over

M in p, but it is also Ma-indiscernible so we have $a_0 \models \{\varphi(x; b_i') : i \in \omega\}$.

This contradicts Kim's lemma for Kim-dividing, showing T has SOP_1.



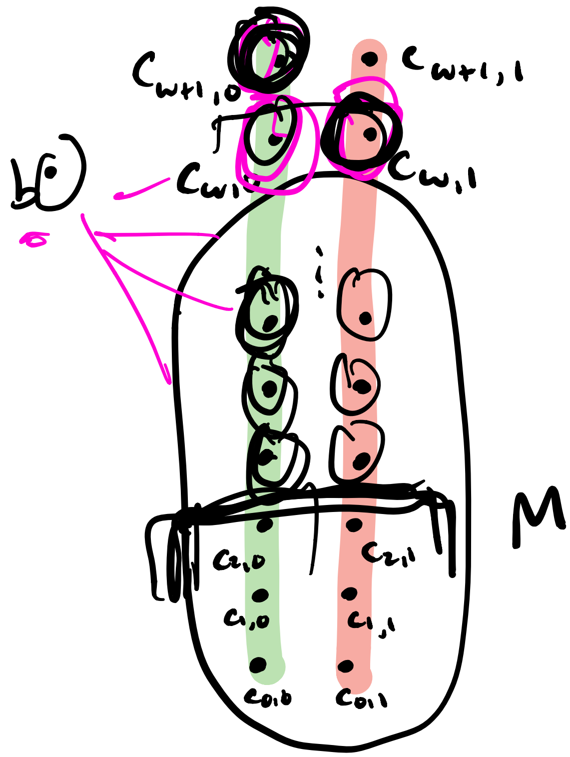
(2) $\Rightarrow$ (1)

Suppose T has SOP_1 via the formula $\varphi(x;y)$. Work, as before, in a Skolemized expansion T^{sk} of T. By compactness and Ramsey, there is an L^{sk}-indiscernible sequence $I = (c_{i,0}, c_{i,1})_{i<\omega+2}$ such that

- $\{\varphi(x;c_{i,0}) : i<\omega+2\}$ is consistent.
- $\{\varphi(x;c_{i,1}) : i<\omega+2\}$ is 2-inconsistent.
- $c_{i,0} \equiv^{L^{sk}}_{c_{<i,0}c_{<i,1}} c_{i,1}$ for all $i<\omega+2$.

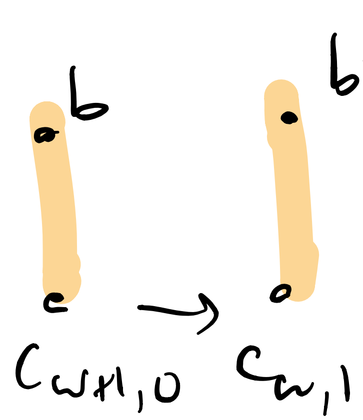
Choose $b \models \{\varphi(x;c_{i,0}) : i<\omega+2\}$. By Ramsey and compactness, we may assume I is b-indiscernible.

Let $M = \mathrm{dcl}_{L^{sk}}(c_{<\omega,0}\, c_{<\omega,1})$.



Note that we have

$$c_{\omega+1,0} \equiv_M c_{\omega,0} \equiv_M c_{\omega,1}.$$



Choose b' such that

$$c_{\omega+1,0}\, b \equiv_M c_{\omega,1}\, b'.$$

By indiscernibility, we have

$$c_{\omega+1,0} \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{M} b, \quad \text{hence}$$

$$c_{\omega,1} \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{M} b'; \quad \text{and also}$$

$$c_{\omega,1} \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{M} c_{\omega+1,0}.$$

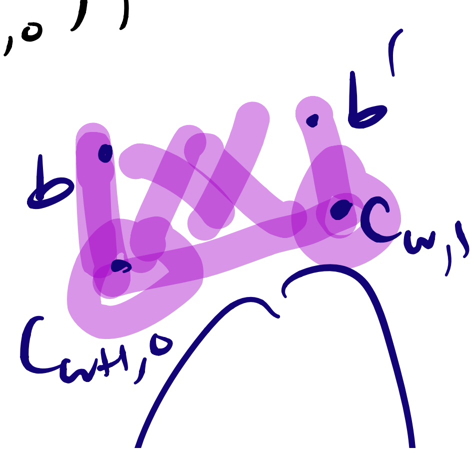
But since $c_{\omega+1,0} \equiv_{c_{<\omega+1,0}\, c_{<\omega+1,1}} c_{\omega+1,1}$

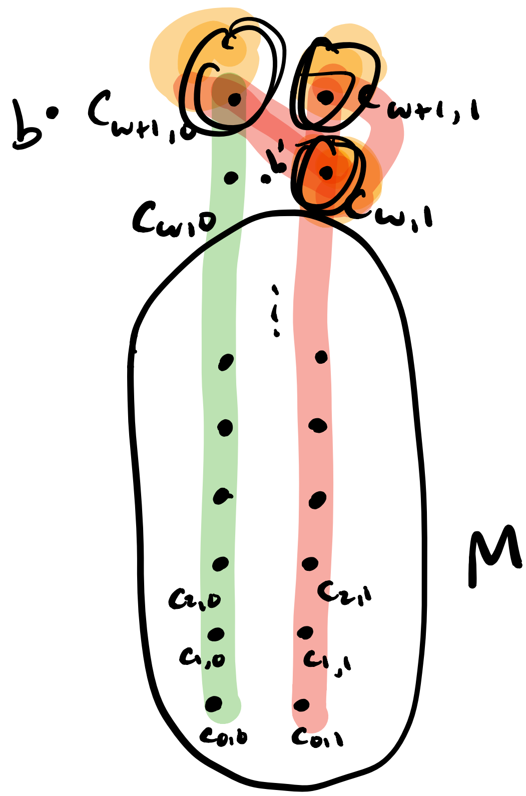
and $\{\varphi(x;c_{\omega,1}), \varphi(x;c_{\omega+1,1})\}$ is

inconsistent, we have

$$\{\varphi(x;c_{\omega,1}), \varphi(x;c_{\omega+1,0})\}$$

is inconsistent.





Since $\varphi(x; c_{\omega+1,0}) \in tp(b/Mc_{\omega+1,0})$

$\varphi(x; c_{\omega,1}) \in tp(b'/Mc_{\omega,1})$,

we see (2) fails. $\blacksquare$

Proof of the Kim-Pillay theorem

First, we show that if $\perp^*$ satisfies (1) – (4), then for any $M \models T$,

$$a \underset{M}{\perp^u} b \Rightarrow a \underset{M}{\perp^*} b.$$

Proof Suppose, towards contradiction, that $a \underset{M}{\perp^u} b$ + $a \underset{M}{\not\perp^*} b$. By strong finite character, there is some $\varphi(x;b) \in tp(a/Mb)$ such that $a' \underset{M}{\not\perp^*} b$ whenever $\models \varphi(a';b)$.

Because $a \underset{M}{\perp^u} b$, there is some $m \in M$ such that $\models \varphi(m;b)$.

Then $m \not\mathop{\downfree}^*_M b$ so, by symmetry,

$b \not\mathop{\downfree}^*_M m$ and, by monotonicity,

$b \not\mathop{\downfree}^*_M M$, contradicting existence. $\blacksquare$

Now suppose $\downfree^*$ additionally satisfies (S).

Suppose we are given $a_0 b_0 \equiv_M a_1 b_1$

with $a_0 \mathop{\downfree}^h_M b_0$ (and hence $a_1 \mathop{\downfree}^{ci}_M b_1$)

and also $b_0 \mathop{\downfree}^h_M b_1$. Then by

the claim (and symmetry), we

have $a_0 \mathop{\downfree}^*_M b_0$, $a_1 \mathop{\downfree}^*_M b_1$, $b_0 \mathop{\downfree}^*_M b_1$

so there is, by the independence theorem,

some a_* with $a_* b_0 \equiv_M a_* b_1 \equiv_M a_0 b_0$. $\blacksquare$

Teaser for next time:

Symmetry in simple theories?

Assume $a \underset{A}{\overset{f}{\perp}} b$. Then

by extn, can generate

seq $(a_i)_{i<\kappa}$ with $a_0 = a$

$a_i \models tp(a/Ab)$ $a_i \underset{A}{\overset{f}{\perp}} a_{<i} b$

By ER, get (indisc) Morley

seq over A, Ab-indisc.

starting w/ a.

Kim's lemma $\Rightarrow$ $b \mathop{\perp}^{f}_{A} a$.

$\Leftarrow$

$$a \mathop{\bigcup}^{K}_{M} b \not\Rightarrow a \mathop{\bigcup}^{i}_{M} b.$$

$\Leftarrow$

$(a_i)_{i<\omega}$ Morley

$$a_{>i} \bigcup a_{\leq i}$$

$(a_i)_{i<\omega}$ $\perp^{K}$-Morley

$a_2 a_3 \nvdash a_0 a_1$