# FORKING AND DIVIDING AT A GENERIC SCALE—AN INTRODUCTION TO KIM-INDEPENDENCE AND NSOP$_1$ THEORIES

**Course Description**: This will be a mini-course concerning Kim-independence and NSOP$_1$ theories. The starting point is simplicity theory, which was developed largely by Kim and Pillay [KP97], drawing on work of Shelah [She80], and the study of important examples, such as pseudo-finite fields [Hru91] and smoothly approximable structures [CH03]. Simplicity theory allowed one to extend the methods of stability theory to understand a number of structures of deep mathematical interest. At its core, simplicity theory is about independence, which is understood via the notion of non-forking. Under the assumption of simplicity, non-forking independence gives a good notion of independence which specializes in examples to give a crucial tool for studying definability in a given structure. After the main results of simplicity theory were established, it was observed that there are a number of structures which admit similarly well-behaved notions of independence but which are nonetheless not simple. The most well-known among these examples were the theory $T^*_{feq}$ of parametrized equivalence relations [DS04], the theory of an infinite dimensional vector space over an algebraically closed field with a non-degenerate bilinear form [Gra99], and the $\omega$-free PAC fields [Cha02]. Developing a theory that would unify and explain the phenomena observed in these concrete structures became a particularly intriguing problem.

This problem was largely solved via the nascent structure theory for NSOP$_1$ theories. The class of NSOP$_1$ theories was isolated by Džamonja and Shelah in [DS04], but this notion went unexplored for quite some time. This changed with [CR16] where, with Artem Chernikov, we introduced a Kim-Pillay criterion for NSOP$_1$ theories and characterized NSOP$_1$ in terms of independent amalgamation of types. This showed that many of the relevant examples were NSOP$_1$ and suggested that independence in these structures could be understood abstractly, on the basis of a general theory. Later, in [KR20], joint with Itay Kaplan, we introduced Kim-independence, which coincides with non-forking independence in simple theories, and corresponds generally to non-forking *at a generic scale*. There, we showed that this notion is symmetric and satisfies the independence theorem, satisfactorily lifting key properties from simplicity theory to a broader context. Subsequent developments deepened the theory, showing, for example, that Kim-independence satisfies local character [KRS19] and is transitive [KR19], and also furnished many interesting new examples [BC18][CK17][KR18]. It is a very active area of neostability theory, and many basic questions remain open.

Here is a rough and tentative plan for the lectures:

(1) **Simplicity theory and Kim's Lemma**: We will give a high-level overview of the results in simplicity theory that will motivate subsequent developments with respect to NSOP$_1$ theories. We will then begin the proof of Kim's lemma for Kim-dividing and show that it characterizes NSOP$_1$ theories.
(2) **Examples and the Kim-Pillay criterion**: We will describe the main method to show a given structure is NSOP$_1$ and use it to build a stock of examples of NSOP$_1$ theories.
(3) **Tree indiscernibles and symmetry**: We will introduce generalized indiscernibles, focusing on the special case of indiscernible trees. We will then give the construction of Morley trees in NSOP$_1$ theories and use it to prove that, in NSOP$_1$ theories, Kim-independence is symmetric.
(4) **The Independence Theorem**: We will prove the independence theorem for Kim-independence in NSOP$_1$ theories.
(5) **PAC fields**: We will take a short detour from NSOP$_1$ to explore the model-theoretic properties of PAC fields. We will show that $\omega$-free PAC fields are NSOP$_1$ and mention the role played by PAC fields in motivating the general theory.
(6) **Further directions**: We will outline some of the further results in the theory of Kim-independence (local character, transitivity, witnessing, Kim's lemma over arbitrary sets) and mention open problems.

## References


[BC18] Silvia Barbina and Enrique Casanovas. Model theory of steiner triple systems. *arXiv preprint arXiv:1805.06767*, 2018.
[CH03] Gregory L Cherlin and Ehud Hrushovski. *Finite structures with few types*. Number 152. Princeton University Press, 2003.
[Cha02] Zoé Chatzidakis. Properties of forking in $\omega$-free pseudo-algebraically closed fields. *The Journal of Symbolic Logic*, 67(03):957–996, 2002.
[CK17] Gabriel Conant and Alex Kruckman. Independence in generic incidence structures. *arXiv preprint arXiv:1709.09626*, 2017.
[CR16] Artem Chernikov and Nicholas Ramsey. On model-theoretic tree properties. *Journal of Mathematical Logic*, page 1650009, 2016.
[DS04] Mirna Džamonja and Saharon Shelah. On $\triangleleft^*$-maximality. *Annals of Pure and Applied Logic*, 125(1):119–158, 2004.
[Gra99] Nicholas Granger. *Stability, simplicity, and the model theory of bilinear forms*. PhD thesis, University of Manchester, 1999.
[Hru91] Ehud Hrushovski. Pseudo-finite fields and related structures. *Model theory and applications*, 11:151–212, 1991.
[KP97] Byunghan Kim and Anand Pillay. Simple theories. *Annals of Pure and Applied Logic*, 88(2-3):149–164, 1997.
[KR18] Alex Kruckman and Nicholas Ramsey. Generic expansion and skolemization in nsop1 theories. *Annals of Pure and Applied Logic*, 169(8):755–774, 2018.
[KR19] Itay Kaplan and Nicholas Ramsey. Transitivity of Kim-independence. *arXiv preprint arXiv:1901.07026*, 2019.
[KR20] Itay Kaplan and Nicholas Ramsey. On Kim-independence. *J. Eur. Math. Soc. (JEMS)*, 22(5):1423–1474, 2020.
[KRS19] Itay Kaplan, Nicholas Ramsey, and Saharon Shelah. Local character of Kim-independence. *Proc. Amer. Math. Soc.*, 147(4):1719–1732, 2019.
[She80] Saharon Shelah. Simple unstable theories. *Annals of Mathematical Logic*, 19(3):177–203, 1980.



*Date*: September 24, 2020; Ramsey.