Combinatorial implication of computability theory

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Introduction

- Many questions in computability theory, even for big question as KL-randomness vs 1−randomness, have close connection to combinatorics.
- We present one example in this talk. We prove that a question of Miller and Solomon——that whether every coloring $c : d^{<\omega} \to k$ admits a c-computable variable word infinite solution, is equivalent to a natural, nontrivial combinatorial question.

We thank Denis Hirschfeldt, Benoit Monin and Ludovic Patey for helpful discussion on the first example.

1. The question of Miller and Solomon
2. Related literature
3. The combinatorial equivalence
4. On $Oppress^d_k(n_0 \cdots n_r)$

VWI problem

We adopt the problem-instance-solution framework to introduce the following problem. We first introduce some notation.

Definition 1 (Variable word)

- An *n-variable word over d* is a sequence v (finite or infinite) of $\{0, \cdots, d-1\} \cup \{x_0, x_1, \cdots\}$ where there are n many variables in v.

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- Given an $\vec{a} \in d^m$, an n-variable word v, suppose $x_{m_0}, x_{m_1}, \cdots, x_{m_{n-1}}$ occur in v with $m_{\hat{n}-1} < m_{\hat{n}}$ for all $\hat{n} < n$. We write $v(\vec{a})$ for the $\{0, \cdots, d-1\}$-string obtained by substitute $x_{m_{\hat{n}}}$ with $\vec{a}(\hat{n})$ in v for all $\hat{n} < m$ and then truncating the result just before the first occurrence of $x_{m_{\hat{n}+1}}$.

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- We write $P_{x_m}(v)$ for the set of positions of x_m in v, namely $\{t : v(t) = x_m\}$; the *first occurrence* of a variable x_m in v refers to the integer $\min P_{x_m}(v)$.

VWI problem

Example 2

Infinite variable word v on $\{0,1\}$:

$$\begin{aligned} & 011 \quad x_0x_0\ 011 \quad x_1\ \ x_0x_0 \quad x_1x_1 00 \quad x_2x_2 \cdots \\ \vec{a} = 10, v(\vec{a}) = & 011 \quad\ \ 11\ 011 \quad\ 0\ \ 11 \quad\quad 0000 \quad\quad \cdots \\ P_{x_0}(v) = & \{3,4 \quad ,9,10,\cdots\}. \end{aligned} \tag{1.1}$$

VWI problem

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& 011 \quad x_0x_0\ 011 \quad x_1 \quad x_0x_0 \quad x_1x_100 \quad x_2x_2\cdots \qquad (1.1)\\
\vec{a} = 10, v(\vec{a}) = & 011 \quad 11\ 011 \quad 0 \quad 11 \quad 0000 \quad \cdots\\
P_{x_0}(v) = & \{3,4\ \ ,9,10,\cdots\}.
\end{aligned}
$$

Definition 3

- Problem: $\mathsf{VWI}(d,k)$.
- Instance: $c : d^{<\omega} \to k$.
- Solution: an ω-variable word v such that $\{v(\vec{a}) : \vec{a} \in d^{<\omega}\}$ is monochromatic.

VWI vs RCA

Joe Miller and Solomon proposed the following question in [Miller and Solomon, 2004].

Question 4

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Or in terms of computability language:

Question 5

Does every $\mathsf{VWI}(d, k)$-instance c admit c-computable solution?

Other versions of variable word problem

Definition 6 (VW, OVW)

If we require the occurrence of x_i being finite for all i then the problem is called VW.
If we require all the occurrence of x_i comes before any occurrence of x_{i+1} then it is called OVW (ordered variable word).

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The problem is proposed by [Carlson and Simpson, 1984] and studied in [Miller and Solomon, 2004] [Liu et al., 2017]. Clearly,

Theorem 7

$\mathsf{VWI}(d,k) \le \mathsf{VW}(d,k) \le \mathsf{OVW}(d,k)$.
$\mathsf{VWI}(d,k) \Leftrightarrow \mathsf{VWI}(d,k+1), \mathsf{VW}(d,k) \Leftrightarrow \mathsf{VW}(d,k+1), \mathsf{OVW}(d,k) \Leftrightarrow \mathsf{OVW}(d,k+1)$.

The complexity of OVW, VW

Theorem 8 ([Miller and Solomon, 2004])

There exists a computable instance of $\mathsf{OVW}(2,2)$ *that does not admit* Δ^0_2 *solution. Thus* $\mathsf{RCA}_0 + \mathsf{WKL}$ *does not prove* $\mathsf{VW}(2,2)$.

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The following result answers a question of [Miller and Solomon, 2004] and [Montalbán, 2011].

Theorem 9 (Monin, Patey, L)

- *For every computable* $\mathsf{OVW}(2,2)$*-instance* c*, every* $\emptyset'$*-PA degree compute a solution to* c.
- *There exists a computable* $\mathsf{OVW}(2,2)$*-instance such that every solution is* $\emptyset'$*-DNC degree.*

Corollary 10 (Monin, Patey, L)

ACA *proves* $\mathsf{OVW}(2,2)$.

Question 11 ([Miller and Solomon, 2004])

Does $\mathsf{OVW}(d,k)$ or $\mathsf{VW}(d,k)$ implies ACA_0 for some l?

A combinatorial equivalence of "VWI(2, 2) vs RCA"

For two sets of numbers A, B, write $A < B$ iff $\max A < \min B$.

Definition 12 ($Oppress(n_0 \cdots n_{r-1})$)

For a finite sequence $n_0, n_1, \cdots, n_{r-1}$ of positive integers, let $N_0 = \{0, \cdots, n_0 - 1\}$, $N_1 = \{n_0, \cdots, n_0 + n_1 - 1\}, \cdots$, $N_{r-1} = \{n_0 + \cdots + n_{r-2}, \cdots, n_0 + \cdots + n_{r-1} - 1\}$, and $N = \cup_{s \leq r-1} N_s$. We write $Oppress^d_k(n_0 n_1 \cdots n_{r-1})$ iff the following is true. There exists a function $f : d^N \to k$ such that for every $s \leq r - 1$, every n_s-variable word v over d of length N, if the first occurrence of variables in v consists of N_s, i.e.,

$$\{\min P_{x_m}(v) : m \in \omega\} = N_s,$$

then there exist $\vec{a}_0, \vec{a}_1 \in d^{n_s}$ such that $f(v(\vec{a}_0)) \neq f(v(\vec{a}_1))$. In that case we say f witnesses $Oppress^d_k(n_0 \cdots n_{r-1})$.

A combinatorial equivalence of "$\mathsf{VWI}(2,2)$ vs RCA"

Theorem 13

The following are equivalent:

- *There exists a* $\mathsf{VWI}(d,k)$*-instance* c *that does not admit* c*-computable solution.*
- *There exists an infinite sequence of positive integers* $n_0 n_1 \cdots$ *such that for all* $r \in \omega$, $Oppress^d_k(n_0 \cdots n_r)$ *holds.*

Intuition on $Oppress_k^d(n_0 \cdots n_{r-1})$

For $\vec{n}, \vec{\hat{n}} \in \omega^{<\omega}$ we write $\vec{n} \le \vec{\hat{n}}$ if $|\vec{n}| = |\vec{\hat{n}}|$ and $\vec{n}(s) \le \vec{\hat{n}}(s)$ for all $s < |\vec{n}|$. We say $\vec{n}$ is a subsequence of $\vec{\hat{n}}$ if there are integers $s_0 < s_1 < \cdots < s_{m-1} < |\vec{\hat{n}}|$ such that $\vec{n} = \vec{\hat{n}}(s_0) \cdots \vec{\hat{n}}(s_{m-1})$. It's obvious that:

Proposition 14

If $\vec{n}$ is a subsequence of $\vec{\hat{n}}$ or $\vec{n} \ge \vec{\hat{n}}$, then $Oppress_k^d(\vec{\hat{n}})$ implies $Oppress_k^d(\vec{n})$.

Intuition on $Oppress^d_k(n_0 \cdots n_{r-1})$

Proposition 15

$Oppress^2_2(22), Oppress^2_2(222)$ holds. $Oppress^2_2(n)$ holds for all $n > 0$.

Proof.

To see $Oppress^2_2(22)$, consider

$$f(\vec{a}) = \vec{a}(0) + \vec{a}(1) + \vec{a}(2) \ mod\ 2.$$

To see $Oppress^2_2(222)$, consider

$$f(\vec{a}) = I(\vec{a}(0) + \vec{a}(1) > 0) + \vec{a}(2) + \vec{a}(3) + \vec{a}(4) \ mod\ 2.$$

Where $I()$ is the indication function. To see $Oppress^2_2(n)$, simply consider $f(\vec{a}) = \vec{a}(0) \ mod\ 2$. □

Intuition on $Oppress_k^d(n_0 \cdots n_{r-1})$

Proposition 16

$Oppress_2^2(2222)$ does not hold.

Proof.

We don't know the proof. Adam P. Goucher at Mathoverflow examined this using SAT solver (https://mathoverflow.net/questions/293112/ramsey-type-theorem). It's easy to check that the following functions don't work:

$$f(\vec{a}) = I(\vec{a}(0) + \vec{a}(1) > 0) + \vec{a}(2) + \vec{a}(3) + \vec{a}(4) + \vec{a}(6) \ mod \ 2; \qquad (3.1)$$

$$f(\vec{a}) = I(\vec{a}(0) + \vec{a}(1) > 0) + I(\vec{a}(2) + \vec{a}(3) > 0) + \\ + \vec{a}(4) + \vec{a}(5) + \vec{a}(6) \ mod \ 2;$$

□

Proof of theorem 13

($\Leftarrow$)

- A Turing functional Ψ^X computes a variable word if Ψ^X is an enumerable set (possibly finite) $\{v_0, v_1, \cdots\}$ of finitely long variable words such that $v_0 \preceq v_1 \preceq \cdots$.

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- Let $X \in Oppress_k^d$.
- Putting priority argument aside, assume each Turing functional is total. i.e.,
  for each $r \in \omega$, let $v_r \in \Psi_r^X$ be such that v_r contains $X(r)$ many variables whose first occurrence is after $|v_{r-1}|$.

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  for each $r \in \omega$, let $v_r \in \Psi_r^X$ be such that v_r contains $X(r)$ many variables whose first occurrence is after $|v_{r-1}|$.
- Suppose $(f_r : r \in \omega)$ witnesses $Oppress_k^d(X \upharpoonright r)$. We transform these f_r to a coloring c so that there is no $v \succeq v_r$ such that $|c(\{v(\vec{a}) : \vec{a} \in d^{n_v}\})| = 1$.

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- To define c on d^n, let $r(n)$ be the maximal integer such that $|v_{r(n)}| \leq n$. We ensure that c on d^n "oppress" v_r for all $r \leq r(n)$.

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- Let P_r be the set of first occurrence of variables in v_r whose first occurrence is after $|v_{r-1}|$. W.l.o.g, suppose $|P_r| = X(r)$ for all $r \in \omega$.
- Define $c(\vec{a}) = f_{r(n)+1}(\vec{a} \restriction \cup_{r \leq r(n)} P_r)$.

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- Take advantage of some particular algorithms $\Phi_0, \Phi_1, \cdots$ and show that their failure (to compute a solution to c) gives rise to a sequence $X \in Oppress_k^d$.

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- $\Phi_0^c, \Phi_1^c, \cdots$ are greedy algorithms in the sense that they extend their current computation (which is a finitely long variable word) whenever possible. More precisely,

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- Φ_{r+1}^c extends its current computation from v_{r+1} to some $\hat{v} \succeq v_{r+1}$ where $\hat{v}$ has more variables than v_{r+1}, whenever it is found that for *some* $\vec{a} \in d^{|v_r|+1}$, $|c(\{\hat{v}(\vec{b})/\vec{a} : \vec{b} \in d^{n_{\hat{v}}}\})| = 1$.

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- Moreover, Φ_{r+1}^c will build its solution v_{r+1} based on $\Phi_0^c, \cdots, \Phi_r^c$ in the sense that all variables in v_{r+1} occur after $|v_r|$ and if some $\Phi_{\tilde{r}}^c$ extends its current computation, then all Φ_r^c (where $r > \tilde{r}$) will restart all over again.

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- there is no $\hat{v} \succeq \hat{v}_r$ such that for some $\vec{a} \in d^{|\hat{v}_{r-1}|}$, $|c(\{\hat{v}(\vec{b})/\vec{a} : \vec{b} \in d^{n_{\hat{v}}}\})| = 1$; moreover, all variables in v_r occur after $|v_{r-1}|$ and $|v_r| > |v_{r-1}|$.
- We show that $n_0 n_1 n_2 \cdots \in Oppress_k^d$.

Proof of theorem 13

- Fix an $r \in \omega$, let $N = n_0 + \cdots + n_r$. To define $f : d^N \to k$ witnessing $Oppress_k^d(n_0 \cdots n_r)$, for every $\vec{a} \in d^N$ we map $\vec{a}$ to a word $\vec{\hat{a}} = h(\vec{a}) \in d^{|\hat{v}_r|}$ and let $f(\vec{a}) = c(\vec{\hat{a}})$.

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- Intuitively, h is defined by connecting each element of N, say $n_0 + \cdots + n_{s-1} + m$, to a set $P_{x_m}(\hat{v}_s)$ and copy the value $\vec{a}(n_0 + \cdots + n_{s-1} + m)$ to $\hat{a}(t)$ for all $t \in P_{x_m}(\hat{v}_s)$. More precisely,

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- Suppose $\vec{a} = \vec{a}_0 \cdots \vec{a}_r$ where $|\vec{a}_s| = n_s$ for all $s \leq r$. Let

$$\vec{\hat{a}}_s = \hat{v}_s(\vec{a}_s) \upharpoonright_{|\hat{v}_{s-1}|}^{|\hat{v}_s|-1} \text{ and } h(\vec{a}) = \vec{\hat{a}}_0 \cdots \vec{\hat{a}}_r.$$

Let $Oppress_k^d$ denote the set of infinite sequence of integers $n_0, n_1, \cdots$ such that $Oppress(n_0 \cdots n_r)$ holds for all $r \in \omega$.

Theorem 17

The following two classes of oracles are equal:

$$\{D \subseteq \omega : D' \text{ computes a member in } Oppress_k^d.\}$$
$$\{D \subseteq \omega : D \text{ computes a } \mathsf{VWI}(d,k)\text{-instance } c \text{ that does not admit a } c\text{-computable solution.}\}$$

Relation to Hales-Jewett theorem

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$$\text{there exists an } N \text{ such that for every } c : d^N \to k,$$
$$\text{there exists an } n\text{-variable word } v \text{ of length } N \text{ such that}$$
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  there exists an N such that for every $c : d^N \to k$,
  there exists an n-variable word v of length N such that
  $|c(\{v(\vec{a}) : \vec{a} \in d^n\})| = 1$.

Theorem 18 (Hales-Jewett theorem [])

For every $d, k, n \in \omega$, $HJ(d, k, n)$ holds.

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- The density HJ theorem $\Rightarrow$ the density van der Waerden theorem, namely Szemerédi's theorem, which asserts that for every set A of integers of positive density (meaning $\limsup_{n\to\infty} |A \cap n|/n > 0$), every $r \in \omega$, there exists an arithmetical progression in A of length r (conjectured by Erdős and Turán).

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- Given d, k, n, the assertion that there exists an r such that $Oppress_k^d(n \underbrace{\cdots}_{r \text{ many}} n)$ does not hold implies $HJ(d, k, n)$. Thus, the following Lemma 19 directly implies Hales-Jewett theorem.

Lemma 19

For every $d, k, n \in \omega$, there exists an r such that $Oppress_k^d(n \underbrace{\cdots}_{r\ many} n)$ does not hold.

Proof.

- For example we prove this for $d, n = 2$.

Lemma 19

For every $d, k, n \in \omega$, *there exists an* r *such that* $Oppress^d_k(n \underbrace{\cdots}_{r\ many} n)$ *does not hold.*

Proof.

- For example we prove this for $d, n = 2$.
- Using $HJ(4, k, 1)$, let r be the witness.
- Show that $Oppress^d_k(2 \underbrace{\cdots}_{r \text{ many}} 2)$ does not hold.

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- Using $HJ(4, k, 1)$, let r be the witness.
- Show that $Oppress_k^d(2 \underbrace{\cdots}_{r \text{ many}} 2)$ does not hold.
- Code 2^{2r} into 4^r where $\vec{a}(2t)\vec{a}(2t+1)$ $(00, 01, 10, 11$ respectively) is coded into $\vec{\hat{a}}(t)$ $(0, 1, 2, 3$ respectively).

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- Code 2^{2r} into 4^r where $\vec{a}(2t)\vec{a}(2t+1)$ $(00, 01, 10, 11$ respectively) is coded into $\vec{\hat{a}}(t)$ $(0, 1, 2, 3$ respectively).
- Given a coloring $c : 2^{2r} \to k$, consider $\hat{c} : 4^r \ni \vec{\hat{a}} \mapsto c(\vec{a})$.
- Let $\hat{v}$ be a 1-variable word monochromatic for $\hat{c}$ and consider v such that $v(2t)v(2t+1) = 00, 01, 10, 11, x_0x_1$ respectively if $\hat{v}(t) = 0, 1, 2, 3, x_0$ respectively.

□

Lemma 20

There exist $n_0 \cdots n_r$ such that $Oppress_2^2(n_0 \cdots n_r)$ holds but $Oppress_2^2(n_0 \cdots n_r n)$ does not hold for all n.

Proof.

For example, $n_0 \cdots n_r = 1$ and note that $Oppress_2^2(1)$ is true but $Oppress_2^2(1n)$ is not true for any n. □

Some open questions

Question 21

Does $Oppress_2^2(2223)$ holds? Does $Oppress_2^2(222n)$ holds for sufficiently large n?

Many thanks

Carlson, T. J. and Simpson, S. G. (1984).
A dual form of Ramsey's theorem.
Adv. in Math., 53(3):265–290.

Liu, L., Monin, B., and Patey, L. (2017).
A computable analysis of variable words theorems.
accepted by Proceedings of the American Mathematical Society.

Miller, J. S. and Solomon, R. (2004).
Effectiveness for infinite variable words and the dual Ramsey theorem.
Arch. Math. Logic, 43(4):543–555.

Montalbán, A. (2011).
Open questions in reverse mathematics.
Bulletin of Symbolic Logic, 17(03):431–454.