# INL (instantial neighborhood logic)
# - tableau, sequent calculus, interpolation

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Outline

- Instantial neighborhood logic INL
- Semantic tableau system TABinl
- Sequent calculus G3inl
- Lyndon interpolation theorem
- Future directions

Abbreviation: “nbd” means “neighborhood”

Instantial neighborhood logic INL

# Nbd structures

- Frame: $\mathfrak{F} = (W, \sigma)$
  - $W \neq \varnothing$, a domain;
  - $\sigma : W \mapsto 2^{2^W}$, a nbd function.
- Possible properties of nbd functions
  - nbd's are non-empty,
  - nbd's of $w$ always contain $w$,
  - each point has exactly 1 nbd
    - degenerates to relational semantics,
  - closure properties...

  None of above assumed for the current work.
- Model: $\mathfrak{M} = (\mathfrak{F}, V)$
  - $\mathfrak{F}$, a nbd frame;
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# Nbd logic

- Basic modal language: unary operator $\Box$.
- Truth definition - a $\exists\forall$ reading of $\Box$:
  - $\mathfrak{M}, w \models \Box\alpha$
    iff
    $(\exists N \in \sigma(w))(\forall n \in N)\, \mathfrak{M}, n \models \alpha$.
  - a nbd (of the current point) has $\alpha$ true everywhere inside.
  - B.t.w., alternative definitions exist.
- Some schemes as examples:

| invalid | valid |
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| $\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta)$ | $\Box(\alpha \wedge \beta) \to \Box\alpha \wedge \Box\beta$ |
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  - 2-sorted, $(j;1)$-ary (where $j \in \mathbb{N}$) operator:

$$\Box(\alpha_1, ..., \alpha_j; \alpha_0)$$

  - Subformulas $\alpha_1, ..., \alpha_j$ are called instances.
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Semantic tableau system TABinl

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- General idea: satisfiability reduction
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  - Reduce goals to subgoals (by rules), extend potential model upon request;
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- Rules for classical propositional logic

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- In order to be satisfied:
  - A $\Box$-formula requires a nbd (of certain type);
    A $\neg\Box$-formula refutes any nbd (of certain type);
    Other formulas govern locally.
  - $\Box$-formulas do not work together to close a goal;
    they each does,
    together with all $\neg\Box$-formulas in the same goal.
- The rule takes from a goal:
  - one $\Box$-formula (with $j$-many instances), and
  - any finite number (say $k$) of $\neg\Box$-formulas
    (resp. with $j_1, \ldots, j_k$-many instances).

$$\Box(\alpha_1, \ldots, \alpha_j; \alpha_0)$$
$$\neg\Box(\beta_1^1, \ldots, \beta_{j_1}^1; \beta_0^1)$$
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    Other formulas govern locally.
  - $\Box$-formulas do not work together to close a goal; they each does,
    together with all $\neg\Box$-formulas in the same goal.
- The rule takes from a goal:
  - one $\Box$-formula (with $j$-many instances), and
  - any finite number (say $k$) of $\neg\Box$-formulas
    (resp. with $j_1, ..., j_k$-many instances).

$$\Box(\alpha_1, ..., \alpha_j; \alpha_0)$$
$$\neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)$$
$$\vdots$$
$$\neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)$$

# Semantic tableau system TABinl

- In order to be satisfied:
  - A $\Box$-formula requires a nbd (of certain type);
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# Semantic tableau system TABinl

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots\\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\Big|\Big|\ \Big|\alpha_0 \wedge \sigma \qquad \Big|_{\sigma\in\{\alpha_x\}^j_{x=1}}\ \Big|\Big|}$$

- $\Box(\alpha_1, ..., \alpha_j; \alpha_0)$ requires a nbd with (generally) $j$ points.
  A nbd is consistent, if all its points are.
- $\forall i \in \{1, ..., k\}$, $\neg\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)$ requires
  either - $\beta^i_0$ fails at some point,
  or - $\beta^i_h$ fails at each point for some $h \in \{1, ..., j_i\}$.
- $j_i + 1$ options for each $i$, so $\prod^k_{z=1}(j_z + 1)$ options in total.
  Index possible nbd's by the option it takes, e.g., $I = \langle I(1), ..., I(k)\rangle$.

# Semantic tableau system TABinl

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- $\forall i \in \{1, ..., k\}$, $\neg\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)$ requires that

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  or - $\beta^i_h$ fails at each point for some $h \in \{1, ..., j_i\}$ - "$I(i) \neq 0$".
- $j_i + 1$ options for each $i$, so $\prod^k_{z=1}(j_z + 1)$ options in total.

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- Each of these options raises a possible nbd to satisfy the goal,

  In order to close, each option has to be closed.
- Each possible nbd is a collection of points (subgoals),

  To close a nbd, it is enough to close one subgoal in it.
- It is destructive,

  No formulas (used or not) above the line remain effective below the line.
- It degenerates to special cases where certain parameters are 0.

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# An example

- Check $\Box(;q) \wedge \Box(p \vee q; \neg\Box(;q)) \rightarrow \Box(p;\top)$ in TABinl.
  - Start with the goal $\neg(\Box(;q) \wedge \Box(p \vee q; \neg\Box(;q)) \rightarrow \Box(p;\top))$
  - Applying propositional rules, the goal is reduced to:

$$\Box(;q)$$
$$\Box(p \vee q; \neg\Box(;q))$$
$$\neg\Box(p;\top)$$

  - No more propositional rules applicable.

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$$\begin{array}{l} \Box(;q) \\ \Box(p \vee q; \neg\Box(;q)) \\ \neg\Box(p; \top) \end{array}$$

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$$\dfrac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\left|\left| \, |\alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,\dots,k\}}^{l(i)\neq 0} \neg\beta^i_{l(i)}|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^y_0\}^{l(y)=0}_{y\in\{1,\dots,k\}}} \right|\right|_{l\in\bigotimes^k_{z=1}\{0,\dots,j_z\}}}$$

- Check $\Box(;q) \wedge \Box(p \vee q; \neg\Box(;q)) \rightarrow \Box(p; \top)$ in TABinl.
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$$\begin{array}{l}\Box(;q)\\ \Box(p \vee q; \neg\Box(;q))\\ \neg\Box(p;\top)\end{array}$$

  - The nbd rule takes exactly 1 $\Box$-formula, there are 2 phases: Phase $\Box(;q)$ and Phase $\Box(p \vee q; \neg\Box(;q))$.

# An example

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\left|\left| \left|\alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,\ldots,k\}}^{l(i)\neq 0} \neg\beta^i_{l(i)}\right|_{\sigma\in\{\alpha_x\}_{x=1}^{j}\cup\{\neg\beta_0^y\}_{y\in\{1,\ldots,k\}}^{l(y)=0}} \right|\right|_{l\in\bigotimes_{z=1}^{k}\{0,\ldots,j_z\}}}$$

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# An example

$$\frac{\Box(;\alpha_0) \qquad \Box(;q)}{\neg\Box(\beta_1^1;\beta_0^1) \qquad \neg\Box(p;\top)}$$

$$\Big|\Big|\ |\alpha_0 \wedge \sigma \wedge \bigwedge^{I(1)\neq 0} \neg\beta^1_{I(1)}|_{\sigma\in\varnothing\cup\{\neg\beta^1_0\}^{I(1)=0}}\Big|\Big|_{I\in\{\langle 0\rangle,\langle 1\rangle\}}$$

- Phase $\Box(;q)$:
  - $j=0, k=1, j_1=1$,
    $I \in \bigotimes_{z=1}^{k}\{0,...,j_z\} = \{\langle 0\rangle, \langle 1\rangle\}$.
  - Option $\langle 0\rangle$ raises $||\,|q \wedge \neg\top|\,||$, which can be closed;
  - Option $\langle 1\rangle$ raises $|| \qquad ||$ (empty nbd), which is open.
- Therefore, this phase is open,
  an empty nbd is constructed.

# An example

$$\frac{\Box(;\alpha_0) \quad \neg\Box(\beta^1_1;\beta^1_0) \qquad \Box(;q) \quad \neg\Box(p;\top)}{\Big|\Big|\ \Big|\alpha_0 \wedge \sigma \wedge \bigwedge^{l(1)\neq 0} \neg\beta^1_{l(1)}\Big|_{\sigma\in\varnothing\cup\{\neg\beta^1_0\}^{l(1)=0}}\Big|\Big|_{l\in\{\langle 0\rangle,\langle 1\rangle\}}}$$

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# An example

$$\frac{\Box(\alpha_1;\alpha_0) \qquad \neg\Box(\beta_1^1;\beta_0^1) \qquad\qquad \Box(p\vee q;\neg\Box(;q)) \qquad \neg\Box(p;\top)}{\Big|\Big|\ \Big|\alpha_0\wedge\sigma\wedge\bigwedge^{l(1)\neq 0}\neg\beta^1_{l(1)}\Big|_{\sigma\in\{\alpha_1\}\cup\{\neg\beta_0^1\}^{l(1)=0}}\Big|\Big|_{l\in\{\langle 0\rangle,\langle 1\rangle\}}}$$

- Phase $\Box(p\vee q;\neg\Box(;q))$:
  - $j=1, k=1, j_1=1,\ l\in\bigotimes_{z=1}^{k}\{0,...,j_z\}=\{\langle 0\rangle,\langle 1\rangle\}$.
  - Option $\langle 0\rangle$ raises $||\,|\neg\Box(;q)\wedge(p\vee q)\,|\,\neg\Box(;q)\wedge\neg\top\,|\,||$, which can be closed (continue on the 2nd point);
  - Option $\langle 1\rangle$ raises $||\,|\neg\Box(;q)\wedge(p\vee q)\wedge\neg p\,|\,||$. Continuing on this finally results in an open goal:

$\neg\Box(;q)$
$q$
$\neg p$

- Therefore, this phase is open,
  a nbd made up by only a $q$-point is constructed.

# An example

$$\frac{\begin{array}{cc}\Box(\alpha_1;\alpha_0) & \Box(p\vee q;\neg\Box(;q)) \\ \neg\Box(\beta_1^1;\beta_0^1) & \neg\Box(p;\top)\end{array}}{\left|\left|\ \left|\alpha_0\wedge\sigma\wedge\bigwedge^{l(1)\neq 0}\neg\beta^1_{l(1)}\right|_{\sigma\in\{\alpha_1\}\cup\{\neg\beta_0^1\}^{l(1)=0}}\right|\right|_{l\in\{\langle 0\rangle,\langle 1\rangle\}}}$$

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- Phase $\Box(p \vee q; \neg\Box(; q))$:
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  - Option $\langle 0\rangle$ raises $|| \mid \neg\Box(; q) \wedge (p \vee q) \mid \neg\Box(; q) \wedge \neg\top \mid ||$, which can be closed (continue on the 2nd point);
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$$\frac{\begin{array}{c}\Box(\alpha_1;\alpha_0)\\ \neg\Box(\beta_1^1;\beta_0^1)\end{array} \qquad \begin{array}{r}\Box(p \vee q; \neg\Box(;q))\\ \neg\Box(p;\top)\end{array}}{\left|\left|\ \left|\alpha_0 \wedge \sigma \wedge \bigwedge^{l(1)\neq 0} \neg\beta^1_{l(1)}\right|_{\sigma\in\{\alpha_1\}\cup\{\neg\beta_0^1\}^{l(1)=0}}\right|\right|_{l\in\{\langle 0\rangle,\langle 1\rangle\}}}$$

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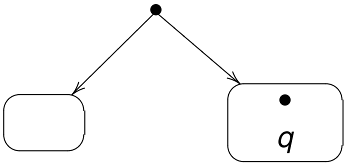
- Check $\Box(;q) \wedge \Box(p \vee q; \neg\Box(;q)) \rightarrow \Box(p;\top)$ in TABinl.
- Since both phases are left open, we see its invalidity, and a counter-model is constructed:

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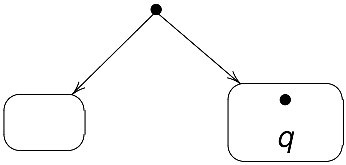


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- TABinl is sound and complete.
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# Sequent calculus G3inl

# A duality

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\left|\left| \left|\alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,\ldots,k\}}^{I(i)\neq 0} \neg\beta^i_{I(i)}\right|_{\sigma\in\{\alpha_x\}_{x=1}^{j}\cup\{\neg\beta_0^y\}_{y\in\{1,\ldots,k\}}^{I(y)=0}} \right|\right|_{I\in\bigotimes_{z=1}^{k}\{0,\ldots,j_z\}}}$$

- Sequent/tableau proofs are usually dual of each other.
  - Sequents $\leftrightsquigarrow$ branches where closure is tested.
  - (Left/right) sides to "$\Rightarrow$" $\leftrightsquigarrow$ (positive/negtive) signs of formulas.
- In the nbd rule of TABinl:
  - a goal is reduced to groups (instead of a group) of subgoals,
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- The dual is a hyper-sequent rule.
  - Hyper-sequents := finite multi-sets of (regular) sequents;
  - like $|\Gamma_1 \Rightarrow \Delta_1|...|\Gamma_n \Rightarrow \Delta_n|$,
    where "|" is read disjunctively.
  - In order to prove a hyper-seq.,
    it is sufficient to prove one of its sequents.

# The hyper-sequent rule

$$\dfrac{\begin{array}{c}\Box(\alpha_1,\dots,\alpha_j;\alpha_0)\\ \neg\Box(\beta^1_1,\dots,\beta^1_{j_1};\beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1,\dots,\beta^k_{j_k};\beta^k_0)\end{array}}{\left\|\ \left|\alpha_0\wedge\sigma\wedge\bigwedge_{i\in\{1,\dots,k\}}^{l(i)\neq 0}\neg\beta^i_{l(i)}\right|_{\sigma\in\{\alpha_x\}_{x=1}^{j}\cup\{\neg\beta^y_0\}_{y\in\{1,\dots,k\}}^{l(y)=0}}\ \right\|_{l\in\bigotimes_{z=1}^{k}\{0,\dots,j_z\}}}$$

$$\dfrac{\left[\begin{array}{c}\left|\alpha_0,\alpha_{-x}\Rightarrow\left\{\beta^i_{l(i)}\right\}_{i\in\{1,\dots,k\}}^{l(i)\neq 0}\right|_{x\in\{-j,\dots,-1\}}\\ \\ \left|\alpha_0\Rightarrow\beta^y_0,\left\{\beta^i_{l(i)}\right\}_{i\in\{1,\dots,k\}}^{l(i)\neq 0}\right|_{y\in\{1,\dots,k\}}^{l(y)=0}\end{array}\right]_{l\in\bigotimes_{i=1}^{k}\{0,1,\dots,j_i\}}}{\Box(\alpha_1,\dots,\alpha_j;\alpha_0)\Rightarrow\{\Box(\beta^i_1,\dots,\beta^i_{j_i};\beta^i_0)\}_{i=1}^{k}}$$

# The hyper-sequent rule

- For a better appearance, let

$$\left( \begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(I)} = \{y \in K \mid I(y) = 0\} & \Omega_K^I = \left\{\beta^i_{I(i)}\right\}_{i \in K}^{I(i) \neq 0} & D = \bigotimes_{i \in K} \{0, 1, ..., j_i\} \end{array} \right)$$

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- Unfold to regular-sequent rules.
  - not genuine ‘hyper’
    - only 1 seq. in the conclusion (in this only nbd rule).
  - specify for each premise (hyper-seq.) with “name” $l$,
    the index (from $J \uplus K^{(l)}$)
    of the regular seq. that exemplifies its provability.
  - unfold to infinitely many regular seq. rules,
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$$\forall l \in D \begin{cases} f(l) = x \in J & \text{if premise l is } \vdash \alpha_0, \alpha_{-x} \Rightarrow \Omega^l_K, \\ f(l) = y \in K^{(l)} & \text{if premise l is } \vdash \alpha_0 \Rightarrow \beta^y_0, \Omega^l_K. \end{cases}$$

We write $f_l$ instead of $f(l)$ for a better appearance...

- $f : D \mapsto J \cup K$ s.t.

(adequacy) $(\forall l \in D)(f_l \in K$ implies $f_l \in K^{(l)}$, e.g., $l(f_l) = 0)$.

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- In multi-set based G3, built-in weakening is necessary.
- This is $\left(\Box^{j,k,f}_{\langle j_1,\dots,j_k\rangle}\right)$, nbd rule with parameters $j,k,j_1,\dots,j_k,f$.
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- G3inl is G3cp (in the language of INL) extended by all $\left(\Box^{j,k,f}_{\langle j_1,\dots,j_k\rangle}\right)$ where $f$ is adequate w.r.t. its other parameters.

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Lyndon interpolation theorem

# Lyndon interpolation of INL

- Lydon interpolation theorem:

  (Let $\mathcal{V}^+(\alpha)/\mathcal{V}^-(\alpha)$ denotes positive/negative atoms in $\alpha$)

  If INL $\vdash \phi \to \psi$, then there is a formula $\theta$ s.t.:
  - $\mathcal{V}^{\pm}(\theta) \subseteq \mathcal{V}^{\pm}(\phi) \cap \mathcal{V}^{\pm}(\psi)$
  - INL $\vdash \phi \to \theta$ and INL $\vdash \theta \to \psi$.

  (a 'polarized' Craig interpolation)
- A general form on sequents:

  If G3inl $\vdash \Pi_L, \Pi_R \Rightarrow \Sigma_L, \Sigma_R$, then there is a formula $\theta$ s.t.:
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  - G3inl $\vdash \Pi_L \Rightarrow \Sigma_L, \theta$ and G3inl $\vdash \theta, \Pi_R \Rightarrow \Sigma_R$.
- The method of splitting sequent calculus (S. Maehara) works.

# Lyndon interpolation of INL

- A splitting version of G3inl
  - Antecedent and succedent of each sequent split into two parts (left | right), like $\Gamma_L|\Gamma_R \Rightarrow \Delta_L|\Delta_R$.
  - For each rule, given an arbitrary splitting conclusion,
    depending on locations of the principals,
    the splitting calculus should offer a rule
    with splitting premises
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# Lyndon interpolation of INL

- For the nbd rule ($\Box^{j,k,f}_{\langle j_1,\ldots,j_k\rangle}$):

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta_0^{f_l}, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{\Pi, \Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in K}, \Sigma}$$

  - Weakening $\Pi, \Sigma$ never matter.
  - Assume the negative principal $\Box(\alpha_1, \ldots, \alpha_j; \alpha_0)$ goes left, the case that it goes right is quite similar.
  - Recall: $J = \{-j, \ldots, -1\}$, $K = \{1, \ldots, k\}$, $D = \bigotimes_{i \in K}\{0, 1, \ldots, j_i\}$.
  - W.l.o.g., assume that positive principals split to $\{\Box(\beta^i_1, \ldots, \beta^i_{j_i})\}_{i \in L} | \{\Box(\beta^i_1, \ldots, \beta^i_{j_i})\}_{i \in R}$ (where $L \uplus R = K$).
  - Also let $D_L = \bigotimes_{i \in L}\{0, 1, \ldots, j_i\}$, $D_R = \bigotimes_{i \in R}\{0, 1, \ldots, j_i\}$. For $l = \langle l(1), \ldots, l(k)\rangle \in D$, let $l_L := l{\restriction}L \in D_L$, and $l_R := l{\restriction}R \in D_R$. Let $\mathbf{0}^R$ be a list of 0's with index-set $R$, similar for $\mathbf{0}^L$. Denote the index-disjoint merge of lists $M$ and $N$ by $M \circ N$.

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# Lyndon interpolation of INL

- When the only negative principal goes left:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0)| \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i \in L} | \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i \in R}}$$

  - For premise with name $l \in D$, choose split premise:
    - $\alpha_0, \alpha_{-f_l}| \Rightarrow \Omega^{l_L}_L | \Omega^{l_R}_R$, if $f_l \in J$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L, \beta^{f_l}_0 | \Omega^{l_R}_R$, if $f_l \in L \subseteq K$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L | \beta^{f_l}_0, \Omega^{l_R}_R$, if $f_l \in R \subseteq K$.

    and denote its assumed interpolant by $\theta_l$.
  - The desired interpolant for the conclusion is

$$\bigvee_{M \in D_L}^{f_{M \circ 0^R} \in R} \Box \left( \left\{ \theta_{M \circ N} \right\}^{f_{M \circ N} \in J \cup L}_{N \in D_R} ; \bigwedge_{N \in D_R}^{f_{M \circ N} \in R} \theta_{M \circ N} \right)$$

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$$\bigvee_{M \in D_L}^{f_{M \circ 0^R} \in R} \Box \left( \left\{ \theta_{M \circ N} \right\}^{f_{M \circ N} \in J \cup L}_{N \in D_R} ; \bigwedge_{N \in D_R}^{f_{M \circ N} \in R} \theta_{M \circ N} \right)$$

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# Lyndon interpolation of INL

- When the only negative principal goes right:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{|\Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in L} | \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in R}}$$

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  - The desired interpolant for the conclusion is

$$\neg \bigvee_{N \in D_R}^{f_{0^L \circ N} \in L} \Box \left( \left\{ \neg \theta_{M \circ N} \right\}^{f_{M \circ N} \in J \cup R}_{M \in D_L} ; \bigwedge_{M \in D_L}^{f_{M \circ N} \in L} \neg \theta_{M \circ N} \right)$$

- Future directions

# Future Directions

- More properties of the nbd function
  - leads to genuine hyper-sequent ?
- Modification on language
  - bound on number of instances
  - infinite language (conjunctions, disjunctions, instances).
- Nabla operator v.s. $\Box(\alpha_1, ..., \alpha_j; \alpha_1 \vee ... \vee \alpha_j)$.
- Uniform interpolation
  - the method of R. Iemhoff seems to work.

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- Thanks !