The bigger picture

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1 Completeness

Recall that a theory T is

- ... *consistent* if there is no sentence σ such that

$$T \vdash \sigma \text{ and } T \vdash \neg\sigma.$$

- ... *complete* if for every sentence σ, exactly one of the following is true:

$$T \vdash \sigma \text{ or } T \vdash \neg\sigma.$$

Equivalently, T is complete if T has a unique model, up to elementary equivalence. Complete theories are useful for proving decidability/computability results:

Example 1.1. *If T is a computably-enumberable complete theory and $M \models T$, then the set*

$$Th(M) = \{\sigma : M \models \sigma\}$$

is decidable, i.e., computable.

This can be used to prove the decidability of $Th(M)$ for many rings and fields appearing in algebra, such as

- The field $\mathbb{C}$ of complex numbers.
- The field $\mathbb{R}$ of real numbers.
- The field $\mathbb{Q}_p$ of p-adic numbers, and the ring $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ of p-adic integers.
- The algebraic closure $\mathbb{F}_p^{alg}$ of the field with p elements.
- The algebraic closure $\mathbb{Q}^{alg}$ of the rational numbers.
- The ring $\mathcal{O} \subseteq \mathbb{Q}^{alg}$ of algebraic integers.

- The rings $\mathbb{C}[[X]]$ and $\mathbb{R}[[X]]$ of formal power series in one variable.

In each case, one writes down a computable theory T, often very complicated[1], and one then verifies

- $M \models T$.
- T is complete. (This usually takes a lot of work)

2 Categoricity and completeness

Definition 2.1. *A theory T is* categorical *(in the absolute sense) if T has exactly one model, up to isomorphism: T has a model M and every model $M' \models T$ is isomorphic $M \cong M'$.*

If T is categorical, then T is complete. However, essentially no theories are categorical, because of the Löwenheim-Skolem theorem:

Theorem 2.2 (Löwenheim-Skolem)**.** *Let T be a countable theory with at least one infinite model. Let κ be an infinite cardinal number.[2] Then there is $M \models T$ with $|M| = \kappa$.*

As a corollary, if T is categorical, the unique model M of T is finite. So absolute categoricity is unfortunately not a very helpful concept. However, a related notion turns out to work correctly. First note that we can rephrase the Löwenheim-Skolem theorem as follows:

Theorem 2.3 (Löwenheim-Skolem, variant form)**.** *Let M be an infinite structure in a countable language. Let κ be an infinite cardinal. Then there is an elementarily equivalent structure $M' \equiv M$ with cardinality $|M'| = \kappa$.*

Then define

Definition 2.4. *A theory T is* κ-categorical *if there is a unique model of cardinality κ, up to isomorphism.*

Using the variant form of Löwenheim-Skolem, one can prove

Theorem 2.5 (Łoś-Vaught criterion)**.** *Let T be a countable theory. If T is κ-categorical, then any two infinite models of T are elementarily equivalent. In particular, if all models of T are infinite, then T is complete.*

[1] For example, for $\mathbb{Z}_p$, the theory says that $\mathbb{Z}_p$ is a valuation ring, the value group is a model of Pressburger arithmetic, the valuation $v(p)$ of p is the minimal positive element of the value group, the residue field has p elements, and the valuation ring is henselian.

[2] More generally, we can let T be uncountable, and then we must take $\kappa \geq |T|$.

3 Examples of κ-categoricity

If K is an algebraically closed field, then K is determined up to isomorphism by two invariants:

- The *characteristic* $\mathrm{char}(K)$, which is the unique number $p \in \{0, 2, 3, 5, 7, \ldots\}$ such that $p\mathbb{Z}$ is the kernel of the unique ring homomorphism

$$\mathbb{Z} \to K.$$

  More concretely, the characteristic determines the minimal subfield $F \leq K$:

  - If $\mathrm{char}(K) = 0$, the minimal subfield is $\mathbb{Q}$.
  - If $\mathrm{char}(K) = p > 0$, the minimal subfield is $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

- The transcendence degree $\mathrm{tr.deg}(K/F)$ of K over the minimal subfield.

For example, up to isomorphism there is a unique field of characteristic 0 and transcendence degree 2, namely

$$\mathbb{Q}(X_1, X_2)^{alg},$$

the algebraic closure of the field of rational functions in two variables.

Moreover, $|K| = \aleph_0 + \mathrm{tr.deg}(K/F)$. From this, one sees that the following theories are κ-categorical for all $\kappa > \aleph_0$:

- ACF_0, the theory of algebraically closed fields of characteristic 0.
- ACF_p, the theory of algebraically closed fields of characteristic p.

The models of these theories are infinite, so it follows that ACF_0 and ACF_p are complete. As a consequence, $Th(\mathbb{C})$ is decidable.

If K is a field, similar methods show that the theory of infinite K-vector spaces[3] is complete and κ-categorical for $\kappa > |K|$. The proof uses dimension of vector spaces instead of transcendence degree.

All these examples are instances of a more general fact:

Proposition 3.1. *If T is strongly minimal and complete, then T is κ-categorical for $\kappa > |T| + \aleph_0$.*

Proof sketch. The proof of this is essentially as follows: fix a monster model $\mathbb{M}$ of T. Let M, M' be two models of T of the same cardinality $\kappa > |T| + \aleph_0$. We may assume $M, M' \preceq \mathbb{M}$. Let B and B' be $\mathrm{acl}(-)$-bases of M and M', respectively. By cardinal arithmetic,

$$|B'| = |M'| = \kappa = |M| = |B|,$$

[3] A model of this theory is a structure $(V, 0, +, \alpha_a : a \in K)$ where V is a K-vector space, $0 \in V$ is zero, $+ : V \times V \to V$ is addition, and for each $a \in K$, the unary operation $\alpha_a : V \to V$ is the map $x \mapsto a \cdot x$. Note that K is fixed as part of the language; K is not a sort in the structure. Note also that the language and theory are usually infinite, and uncountable if $|K| > \aleph_0$.

so we can find a bijection $f : B \to B'$. Using Lemma 1.3 in Wednesday's notes, one shows that f is a partial elementary map. As $\mathbb{M}$ is a monster, f extends to an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$. Then

$$f(M) = f(\mathrm{acl}(B)) = \mathrm{acl}(f(B)) = \mathrm{acl}(B') = M',$$

so $M \cong M'$. □

In combination with the Łoś-Vaught criterion, Proposition 3.1 tells us *nothing*, since we assumed T was complete. Nevertheless, Proposition 3.1 turns out to be important for understanding κ-categoricity abstractly.[4]

So there are several examples of theories which are κ-categorical for all $\kappa > \aleph_0$. It turns out there are also several theories which are $\aleph_0$-categorical, but not κ-categorical for any $\kappa > \aleph_0$. The most notable examples are

- DLO, the theory of dense linear orders without endpoints. The unique countable model is $(\mathbb{Q}, \leq)$.
- The theory T of the Rado graph.
- The theory of atomless boolean algebras. See §A below for the definition of Boolean algebras. A boolean algebra B is atomless if for every $0 < x \in B$ there is $y \in B$ with $0 < y < x$.

The vast majority of complete theories arising in practice are not κ-categorical for any κ. For example, if K is an infinite field and $Th(K)$ is κ-categorical for some κ, then $K \models ACF$.

4 Morley's theorem

Łoś conjectured, and Morley proved

Theorem 4.1 (Morley's theorem)**.** *Let T be a countable theory. If T is κ-categorical for some $\kappa > \aleph_0$, then T is κ-categorical for all $\kappa > \aleph_0$.*

Consequently, one can talk about *uncountably categorical* theories—theories that are κ-categorical for any/every uncountable κ.

For Morley's theorem, the journey is more interesting than the destination: the statement of Morley's theorem is useless and boring, but the proof develops many of the central tools of modern model theory. We sketch some highlights of the proof in the next few sections, following the proof in the final chapter of Hodges' *Shorter Model Theory.*

Here is the brief summary of the proof. Let T be κ-categorical for some $\kappa > \aleph_0$. As in the case of strongly minimal theories, we want to associate a "rank" to any model $M \models T$, such that

[4] As in the proof of Theorem 7.4 below.

- The rank of M determines M up to isomorphism.
- The cardinality of M is $\aleph_0$ plus the rank of M.

This will then ensure that M is κ-categorical for *all* uncountable κ.

In fact, one directly reduces to the case of strongly minimal theories: every uncountably categorical theory is "governed" by a strongly minimal theory, in some precise sense.

5 Totally transcendental theories

Let $\mathbb{M}$ be a monster model. For $X \subseteq \mathbb{M}^n$ a definable set, recursively define *Morley rank* $RM(X)$ as follows:

- $RM(X) \geq 0$ iff $X \neq \emptyset$.
- $RM(X) \geq \alpha + 1$ iff there exist definable subsets $X_1, X_2, X_3, \ldots \subseteq X$, pairwise disjoint, with $RM(X_i) \geq \alpha$ for each $i < \omega$.
- If α is a limit ordinal, then

$$RM(X) \geq \alpha \iff (\forall \beta < \alpha : RM(X) \geq \beta)$$

Then define the Morley rank to be

$$RM(X) = \max\{\alpha : RM(X) \geq \alpha\},$$

or $+\infty$ if there is no maximum, or $-\infty$ if $X = \emptyset$.

This definition looks odd, but is closely related to the Cantor-Bendixson theorem. See the Appendix.

If $-\infty < RM(X) < \infty$, there is a maximum value $d < \omega$ such that we can find $X_1, \ldots, X_d$ disjoint definable subsets of X, with $RM(X_i) = RM(X)$. The maximum such d is always finite, and is called the *Morley degree* of X, often denoted $dM(X)$.

Aside 5.1. *If M is not a monster model, one can still define $RM(X)$ and $dM(X)$, but one must pass to elementary extensions. For example, $RM(X) \geq \alpha + 1$ iff there is an elementary extension $M' \succeq M$ and M'-definable disjoint subsets $X_1, X_2, \ldots \subseteq X(M')$ with $RM(X_i) \geq \alpha$. It turns out that this definition agrees with the earlier definition when M is a monster model, and ensures that Morley rank is preserved in elementary extensions:*

Specifically, if $M \preceq M'$ and $\phi(\vec{x}; \vec{a})$ is a formula over M, then the Morley rank of $\phi(M; \vec{a})$ in M equals the Morley rank of $\phi(M'; \vec{a})$ in M'.

Definition 5.2. *A structure M is* totally transcendental[5] *if $RM(X) < \infty$ for all definable sets X.*

[5]This name is terrible and I have no idea where it comes from. We're stuck with it now, though.

This is really a property of the theory. For countable theories, there is an alternate criterion:

Definition 5.3. *A theory T is* $\aleph_0$-stable *if for every countable model $M \models T$, the space of n-types $S_n(M)$ over M is countable.*

Fact 5.4. *If T is a countable complete theory, then T is totally transcendental if and only if T is $\aleph_0$-stable.*

This is closely related to the Cantor-Bendixson theorem. See the appendix.

The first step of Morley's theorem is the following

Proposition 5.5. *Let T be a countable complete theory. Suppose T is κ-categorical for some $\kappa > \aleph_0$. Then T is totally transcendental.*

Proof sketch. Otherwise, by Fact 5.4, there is a countable model M_0 such that $S_n(M_0)$ is uncountable. Pick $\aleph_1$-many distinct types in M_0, and realize them in an elementary extension $M_1 \succeq M_0$ with $|M_1| = \aleph_1$. As $\kappa \geq \aleph_1$, we can find $M_2 \succeq M_1 \succeq M_0$ with $|M_2| = \kappa$. In the model M_2, there is a countable subset $M_0 \subseteq M_2$ over which uncountably many n-types are realized.

Meanwhile,

Lemma 5.6 (Ehrenfeucht-Mostowksi)**.** *If T is a consistent countable theory and κ is a cardinal, there is a model $M \models T$ of size κ with the property that for any countable set $A \subseteq M$, only countably many types over A are realized in the structure M.*

The proof of this lemma is a clever argument using Skolemization and indiscernible sequences—see the final chapter of Hodges' *Shorter Model Theory*.

Anyways, the Lemma produces a model M' of size κ, in which not many types are realized over any countable set. So $M' \not\cong M_2$, contradicting κ-categoricity. $\square$

6 Morley rank and strongly minimality

Morley rank and degree agree with the "dimension" and "degree" we defined for strongly minimal theories.

Fact 6.1. *If $\mathbb{M}$ is strongly minimal, then for every interpretable set X,*

$$RM(X) = \dim(X) < \omega$$
$$dM(X) = \deg(X)$$

In fact, the "right" way to define dimension and degree in strongly minimal structures is to define Morley rank and Morley degree, and then to prove that Morley rank agrees with the dimension from the pregeometry.

Morley rank and degree are related to strong minimality in a second way:

Fact 6.2. *A structure M is strongly minimal iff $RM(M) = dM(M) = 1$.*

More generally, a definable set *$X \subseteq M^n$ is strongly minimal iff $RM(X) = dM(X) = 1$.*

Here, a definable set X is *minimal*[6] if X is infinite and X is not a disjoint union of two infinite definable sets. A definable set X is *strongly minimal* if X remains minimal in elementary extensions. A structure M is minimal (resp. strongly minimal) iff the definable set $M \subseteq M^1$ is minimal (resp. strongly minimal).

Strongly minimal sets are prevalent in totally transcendental theories:

Fact 6.3. *If $\mathbb{M}$ is a totally transcendental monster model, then every infinite definable set X contains a strongly minimal definable subset $Y \subseteq X$.*

To prove this, one takes definable infinite $Y \subseteq X$ minimizing $(RM(Y), dM(Y))$, and verifies that $RM(Y) = dM(Y) = 1$.

7 The proof idea of Morley's theorem

Definition 7.1. *A model $M \models T$ is* prime *over a subset $A \subseteq M$ if for every model M' and partial elementary map*[7]*, there is an elementary embedding of M into M' making the diagram commute*

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & M \arrow[d] \\ & M'. \end{tikzcd}$$

Fact 7.2. *Let $\mathbb{M}$ be a totally transcendental monster model. For every small set $A \subseteq \mathbb{M}$, there is a small model $M \preceq \mathbb{M}$ which is prime over A. Moreover, M is unique up to isomorphism over A.*

Definition 7.3. *A theory T has* Vaughtian pairs *if there is an elementary extension $M \preceq M'$ and a formula $\phi(\vec{x}, \vec{a})$ over M such that*

$$M \neq M'$$
$$\phi(M; \vec{a}) = \phi(M'; \vec{a})$$

and the set $\phi(M; \vec{a})$ is infinite.

Theorem 7.4. *Let T be a countable complete theory. The following are equivalent:*

1. *T is κ-categorical for some $\kappa > \aleph_0$.*

2. *T is totally transcendental and has no Vaughtian pairs.*

3. *T is κ-categorical for all $\kappa > \aleph_0$.*

[6] As I understand it, this is the true origin of the term "minimal" in the name "strongly minimal theories."

[7] See Definition 8.3 in Tuesday's notes.

Proof sketch. For $1 \implies 2$, we have already seen that T is totally transcendental (Proposition 5.5). Suppose T is totally transcendental and has a Vaughtian pair

$$M_1 \preceq M_2$$
$$M_1 \neq M_2$$
$$\phi(M_1; \vec{a}) = \phi(M_2; \vec{a}).$$

Then one can arrange for M_1, M_2 to be countable, using downwards Löwenheim-Skolem. It turns out that in totally transcendental theories, one can "stretch" Vaughtian pairs, producing an arbitrarily long elementary chain

$$M_1 \preceq M_2 \preceq \cdots \preceq M_\omega \preceq \cdots$$

such that for all α,

$$\phi(M; \vec{a}) = \phi(M_\alpha; \vec{a})$$
$$M_{\alpha+1} \neq M_\alpha.$$

After κ steps, one gets M_κ of size κ, with

$$|\phi(M_\kappa; \vec{a})| = |\phi(M_1; \vec{a})| = \aleph_0 < \kappa$$

On the other hand, one can find a model of size κ in which every infinite definable set has size κ, so κ-categoricity must fail. This proves $1 \implies 2$.

Now suppose 2 holds: T is totally transcendental with no Vaughtian pairs. Using Fact 6.3, one can find a formula $\phi(\vec{x})$ such that the set $\phi(\mathbb{M})$ is strongly minimal.[8] Now given any small model M, we may assume $M \preceq \mathbb{M}$. There is a pregeometry on $\phi(M)$ induced by $\operatorname{acl}(-)$. Let B be a basis for this pregeometry. It turns out that...

- M is a prime model over B. Otherwise, if M' is the actual prime model over B, one gets $M \preceq M'$, $M \neq M'$, and $\phi(M) = \phi(M')$, contradicting no vaughtian pairs. (This takes some work.)

- If B and B' are two $\operatorname{acl}(-)$-independent sets in $\phi(\mathbb{M})$ of the same size, and $f : B \to B'$ is a bijection, then f is a partial elementary map. (This takes some work, and is similar to Lemma 1.3 in Wednesday's notes.) Consequently, there is an automorphism $\sigma \in \operatorname{Aut}(\mathbb{M})$ sending B to B'.

[8] This isn't quite right. The formula $\phi(\vec{x})$ actually should have parameters from the prime model M_0 over $\emptyset$. Fact 6.3 could involve parameters from outside M_0. However, using No Vaughtian Pairs, one can prove that $\exists^\infty$ is eliminated. This then allows one to prove that any minimal set in M_0 is strongly minimal. Minimal sets exist in M_0 by a Cantor-Bendixson argument. In the remainder of the proof of Theorem 7.4, we should really work over the prime model M_0, not over $\emptyset$. For example, the pregeometry rank should probably be with respect to $\operatorname{acl}(- \cup M_0)$, and the automorphism should be in $\operatorname{Aut}(\mathbb{M}/M_0)$.

- The size of B is a complete isomorphism invariant for M: if M' is another model and B' is an acl-basis for $\phi(M')$, then
$$|B| = |B'| \implies M \cong M'.$$
Indeed, if $|B| = |B'|$, we can find $\sigma \in \mathrm{Aut}(\mathbb{M})$ sending $\sigma(B) = B'$. Then $\sigma(M)$ is a prime model over B', so
$$M \cong \sigma(M) \cong M'.$$

As in the proof of Proposition 3.1, it follows that T is κ-categorical for all $\kappa > \aleph_0$. □

So in some sense, an uncountably categorical structure M is "governed" by a strongly minimal structure $\phi(M)$ interpretable in M.

8 Stability

Lurking in the background of Morley's theorem is the notion of *stability*:

Definition 8.1. *A theory T is κ-stable if for every model M and $n \geq 1$,*
$$|M| \leq \kappa \implies |S_n(M)| \leq \kappa,$$
where $S_n(M)$ is the space of (complete) n-types over M.

We encountered $\aleph_0$-stability earlier (Definition 5.3).

Definition 8.2. *Fix a theory T. A formula $\phi(\vec{x};\vec{y})$ is* unstable *if there is a model $M \models T$ and elements*
$$\vec{a}_1, \vec{a}_2, \ldots$$
$$\vec{b}_1, \vec{b}_2, \ldots$$
in M such that for every $i, j < \omega$,
$$i < j \iff M \models \phi(\vec{a}_i, \vec{b}_j)$$
Otherwise, we say that $\phi(\vec{x}, \vec{y})$ is stable*.*

Both κ-stability and stability of formulas look very unenlightening. The two concepts are related by the following non-trivial theorem:

Theorem 8.3. *The following are equivalent for a theory T:*

- *T is κ-stable for some infinite cardinal $\kappa \geq |T|$.*
- *Every formula $\phi(x; y)$ is stable.*

We say that a theory T is ***stable*** *if it satisfies these equivalent conditions.*

Stability can be defined in several other equivalent ways.[9]

[9] For example, T is stable iff all types are definable. T is stable iff all indiscernible sequences are totally indiscernible. T is stable iff T is NSOP and NIP. T is stable iff for every formula ϕ, the boolean algebra generated by sets of the form $\phi(\mathbb{M}; \vec{a})$ has Cantor-Bendixson rank less than ∞.

9 Shelah's classification theory

In spite of its odd definition, stability is central to much of modern model theory. Initial research on stability was driven by Shelah's program of Classification Theory, which we now describe.

Given a countable, complete theory T and an uncountable[10] cardinal κ, let $f_T(\kappa)$ be the number of models of T of size κ, counted up to isomorphism. The function $f_T(-)$ is called the *spectrum* of T.

Example 9.1. *If T is uncountably categorical, then f_T is the constant function 1.*

Example 9.2. *Let T be the theory of 2-sorted structures (X, Y) where X, Y are infinite and there is no further structure. This theory is complete—it is $\aleph_0$-categorical. The spectrum f_T is given by*

$$f_T(\aleph_\alpha) = 2 \cdot |\alpha| + 1.$$

For example, there are five models of size $\aleph_2$, namely

$$(\aleph_0, \aleph_2), (\aleph_1, \aleph_2), (\aleph_2, \aleph_2), (\aleph_2, \aleph_1), (\aleph_2, \aleph_0).$$

Shelah's program of Classification Theory proposed to classify theories according to their spectra. This classification was completed by Hart, Hrushovski, and Laskowski. See Wikipedia for a list of possible spectra.

Stability enters the picture because of the following theorem

Theorem 9.3 (Shelah)**.** *If T is unstable, then $f_T(\kappa) = 2^\kappa$ for all uncountable κ.*

So, from the point of view of classification theory, the only interesting theories are stable theories.

10 Examples of stable theories

It turns out that the following theories and structures are stable:

- Any finite structure.
- Any strongly minimal theory/structure.
- More generally, any uncountably categorical theory/structure.
- More generally, any totally transcendental theory/structure.
- T^{eq}, for any stable theory T.

[10]It would be more natural to let κ range over infinite cardinals, but some odd things happen when $\kappa = \aleph_0$, related to Vaught's conjecture.

- The infinite set with no structure.
- The theory ACF of algebraically closed fields.
- More generally, the theory SCF of separably closed fields.
- The theory DCF_0 of *differentially closed fields of characteristic 0.* A differential field is a field $(K, +, \cdot, \partial)$ with an operator $\partial : K \to K$ satisfying the usual addition and multiplication rules

$$\partial(x \cdot y) = x\partial y + y\partial x$$
$$\partial(x + y) = \partial x + \partial y.$$

  For example, the field K of germs of complex-valued meromorphic functions at 0 is a differential field.[11] A *differentially closed field* is an existentially closed differential field. These turn out to be the models of a theory DCF_0, which is stable, and even totally transcendental.

- The theory of abelian groups.
- More generally, the theory of R-modules for any ring R.
- The free non-abelian group on n generators, by deep work of Sela.

Of these theories, the most important is probably DCF_0. The study of stability and totally transcendental theories has offered new insights into the algebraic properties of differential fields.

11 Groups of finite Morley rank

In class, especially on Friday, we proved several statements about groups and fields interpretable in strongly minimal theories.

All these results generalize to groups and fields of finite Morley rank. See Poizat's book *Stable Groups* for the general treatment. For example, Macintyre's theorem really says that any field K of finite Morley rank is algebraically closed.[12]

In fact, groups of finite Morley rank essentially reduce to groups in strongly minimal theories:

Fact 11.1. *If $(G, \cdot)$ is a group of finite Morley rank, there is a chain*

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

of definable subgroups, such that each quotient G_{i+1}/G_i is a group interpretable in a strongly minimal theory. More specifically, there is a strongly minimal subset $X_i \subseteq G_{i+1}/G_i$ which generates G_{i+1}/G_i as a group.

[11] Conversely, any countable differential field of characteristic 0 embeds into K, so the axioms of differential fields are somehow reasonable.

[12] Actually, Macintyre proved that *any* totally transcendental field is algebraically closed.

(cf Theorem 2.13 in Poizat's *Stable Groups*, especially the proof. The key ingredient is Zilber's theorem on indecomposable sets, which holds in a finite Morley rank setting. See Theorem 2.9 in Poizat.)

This fact ensures that simple groups of finite Morley rank are equivalent to simple groups interpretable in strongly minimal theories.[13]

More importantly, this fact ensures that there is a good dimension theory, by the following general nonsense:

Fact 11.2. *Let M be a totally transcendental structure. Let $X_1, \ldots, X_n$ be strongly minimal sets in M. Suppose that M is "governed" by the X_i, in the sense that for any $M'' \succeq M' \succeq M$, if $M'' \neq M'$, then $X_i(M'') \neq X_i(M')$ for some i. Then M has a good dimension theory:*

1. *Morley rank satisfies all the additivity properties (Lemma 2.3, Theorem 2.4 in Wednesday's notes).*
2. *Morley rank is definable in families (as in Theorem 2.5 = Theorem 7.3 in Wednesday's notes).*

For a proof, see Corollary 2.14 in Poizat's *Stable Groups.*

Remark 11.3. *Fact 11.2 also shows that uncountably categorical theories have a good dimension theory.*

The study of groups of finite Morley rank is motivated in part by the example of DCF_0. If G is a "differential algebraic group,"[14] then G is often a group of finite Morley rank, and the facts about groups of finite Morley rank apply to G.

12 Beyond stability

Unfortunately, many algebraic structures of interest fail to be stable. In fact, we can split stability into two related concepts.

Definition 12.1. *Let T be a theory.*

1. *A formula $\phi(\vec{x};\vec{y})$ has the* strict order property *if there is a model $M \models T$ and a sequence $\vec{b}_1, \vec{b}_2, \ldots$ from M such that*

$$\phi(M;\vec{b}_1) \subsetneq \phi(M;\vec{b}_2) \subsetneq \phi(M;\vec{b}_3) \subsetneq \cdots$$

2. *T is* NSOP *iff no formula has the strict order property.*

[13] Consequently, the version of the Cherlin-Zilber conjecture given in class (Conjecture 5.4 in Friday's notes) agrees with the usual version: simple groups of finite Morley rank are algebraic.

[14] Such as $G = \{x \in K : \partial\partial x = 0\}$ with addition. There are more interesting examples of differential algebraic groups coming from the Manin connection on a moving family of abelian varieties.

3. *A formula* $\phi(\vec{x};\vec{y})$ *has the* independence property *if there is a model* $M \models T$ *and elements* $\{\vec{a}_i\}_{i\in\mathbb{N}}$, $\{\vec{b}_S\}_{S\subseteq\mathbb{N}}$ *such that for every* $i \in \mathbb{N}$ *and* $S \subseteq \mathbb{N}$,

$$i \in S \iff M \models \phi(\vec{a}_i;\vec{b}_S).$$

4. T *is* NIP *iff no formula has the independence property.*

Then it turns out that

Fact 12.2. *T is stable iff T is both NIP and NSOP.*

Most of the structures from algebra which admit QE results, such as $\mathbb{R}$ and $\mathbb{Q}_p$, fail to be NSOP, because of the following general problem:

Fact 12.3. *If $M \models T$ is NSOP and $(P,\leq)$ is an interpretable poset in M, then there is a finite number $n < \omega$ such that P contains no chains of length n.*

For example, $\mathbb{R}$ cannot be NSOP because $(\mathbb{R},\leq)$ contains chains of unbounded length. In the field $\mathbb{Q}_p$ of p-adic numbers, the value group

$$\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \cong \mathbb{Z}$$

is an interpretable infinite totally ordered set, so $\mathbb{Q}_p$ cannot be NSOP.

On the other hand, all of the following structures are NIP:

- Any totally ordered set $(X,\leq)$ or totally ordered abelian group $(X,+,\leq)$.
- The fields $\mathbb{R}$ of real numbers, $\mathbb{Q}_p$ of p-adic numbers, and their finite extensions.
- The exponential field $(\mathbb{R},+,\cdot,\exp)$.
- An algebraically closed field $(K,+,\cdot,\mathcal{O})$ expanded with a unary predicate for a valuation ring $\mathcal{O} \subseteq K$.
- The rings of formal power series

  $$\mathbb{C}[[X]], \mathbb{R}[[X]], \mathbb{Q}_p[[X]]$$

  and the fields of formal Laurent series

  $$\mathbb{C}((X)), \mathbb{R}((X)), \mathbb{Q}_p((X))$$

So there has been a lot of work put into generalizing the methods of stability theory to NIP theories.

There are several other generalizations of stability, such as *simple* theories and NTP_2 theories. Gabe Conant has made a handy map of these notions.[15]

[15] For a good reference for NIP theories, see Pierre Simon's *Guide to NIP theories.* For a brief introduction to simple theories, see Grossberg, Iovino, and Lessman's *Primer of Simple Theories.*

13 O-minimality

Definition 13.1. *An ordered structure* $(M, \leq, \ldots)$ *is* o-minimal *if every definable subset* $X \subseteq M$ *is a finite union of points* $\{a\}$ *and open intervals* $(a, b) \subseteq M$*.*

This definition is meant to be analogous to the definition of (strong) minimality: we are assuming something about definable subsets $X \subseteq M^n$ for $n = 1$.

Unlike the situation with (strong) minimality, one has

Theorem 13.2. *If* $M \equiv M'$ *and* M *is o-minimal, then* M' *is o-minimal.*

So there is no need to talk about "strong" o-minimality—it would agree with plain o-minimality.

There are two important things to know about o-minimality, which become remarkable when taken together:

1. The real field $(\mathbb{R}, +, \cdot)$ is o-minimal, and many expansions of $\mathbb{R}$ are also o-minimal, such as

$$(\mathbb{R}, +, \cdot, \exp)$$
$$(\mathbb{R}, +, \cdot, \exp, \tan^{-1})$$

   In fact, there is an o-minimal expansion $\mathbb{R}_{an,pfaff}$ of $\mathbb{R}$ in which

   - For every definable function $f : \mathbb{R} \to \mathbb{R}$, the integral $\int f : \mathbb{R} \to \mathbb{R}$ is also definable.
   - For every analytic or meromorphic function $f : \mathbb{R} \to \mathbb{R}$, the retriction $f \restriction [0, 1] : [0, 1] \to \mathbb{R}$ is definable.

2. O-minimal structures are *extremely* well-behaved and tame. For example,

   - Definable closure and algebraic closure agree, and satisfy exchange, giving a pregeometry.
   - There is a resulting dimension theory on definable sets, satisfying all the conditions of Theorems 2.3-2.5 in Wednesday's notes.
   - Elimination of imaginaries holds.[16]
   - Every definable function $f : M^n \to M^m$ is piecewise continuous. If M is an expansion of a field, then f is piecewise differentiable.
   - If M is $(\mathbb{R}, \leq, \ldots)$, then every definable set X has finitely many connected components, each of which is path connected. If X is compact, then X is homeomorphic to a finite simplicial complex.

The canonical reference for o-minimal theories, especially point 2, is van den Dries' book *Tame topology and o-minimal structures.*

O-minimal theories are NIP but never stable. They can (arguably) be seen as the analogue of strongly minimal theories in the NIP setting.

[16]. . . at least if we're working in an expansion of real closed fields. See `https://arxiv.org/abs/1404.3175` for an exotic counterexample.

14 Geometries and pregeometries

We now take a brief digression into combinatorics and pregeometries.

Definition 14.1. *A pregeometry* (X, cl) *is a* geometry *if* $cl(\emptyset) = \emptyset$ *and for every* $a \in X$, $cl(\{a\}) = \{a\}$.

Given any pregeometry (X, cl), we can construct an associated geometry (Y, cl) as follows:

- Let X_0 be $cl(\emptyset)$. Let $X_1 = X \setminus X_0$.
- On X_1, let $\sim$ be the equivalence relation.

$$a \sim b \iff cl(\{a\}) = cl(\{b\}).$$

- Let Y be $X_1/\sim$. There is an induced closure operation cl on Y making (Y, cl) into a geometry. It is related to the original pregeometry (X, cl) as follows. Let $f : X_1 \to Y$ be the quotient map

$$X_1 \twoheadrightarrow X_1/\sim \cong Y.$$

  Then for any $a \in X$ and $S \subseteq X$,

$$a \in cl(S) \iff f(a) \in cl(f(S \setminus X_0)).$$

Here are two of the motivating examples that explain the term "geometry:"

1. If K is a field, the n-dimensional *affine geometry* over K is the geometry on K^n whose closed sets are
   - the empty set
   - translates $a + V$ of K-linear subspaces $V \leq K^n$.
2. If K is a field, the n-dimensional *projective geometry* over K is the geometry whose underlying set is the n-dimensional projective space over K

$$\mathbb{P}^n(K) = (K^{n+1} \setminus \{\vec{0}\})/(K^\times),$$

   and whose closed sets are the projectivizations of linear subspces $V \subseteq K^{n+1}$:

$$(V \setminus \{\vec{0}\})/(K^\times).$$

   Note that the n-dimensional projective geometry over K is exactly the geometry associated to the linear-independence pregeometry on K^{n+1} (in which the closed sets are the K-linear subspaces of K^{n+1}).[17]

[17] There is also a more explicit way to view projective n-space over K as affine n-space over K with an attached "hyperplane at infinity." For example, the projective line is the affine line plus a point at infinity. The projective plane is the affine plane plus a (projective) line at infinity. Projective 3-space is obtained from affine 3-space by adding a plane at infinity.

Both of these examples generalize to the case where K is a skew field[18], i.e., an associative non-commutative unital ring $(K, +, \cdot, 1)$ in which every non-zero $a \in K$ has a two-sided inverse, i.e., an element a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

The most famous skew field is Hamilton's *quaternions*. Wedderburn's Little theorem says that finite skew fields are commutative (i.e., finite skew fields are fields).

15 Modular geometries and division rings

Definition 15.1. *A pregeometry (X, cl) is* modular *if for any two closed sets $C_1, C_2 \subseteq X$ of finite rank,*

$$rank(C_1 \cup C_2) = rank(C_1) + rank(C_2) - rank(C_1 \cap C_2).$$

The term "modular" is a reference to modular lattices.[19]

A pregeometry M is modular iff the associated geometry is modular.

Given a modular geometry (X, cl), say that $a, b \in X$ are "connected" if $a = b$ or $cl(\{a, b\}) \supsetneq \{a, b\}$. This turns out to be an equivalence relation. Every modular geometry breaks into its connected components, which don't really interact with each other.

The connected modular geometries are classified by the following non-trivial theorem

Theorem 15.2 (Synthetic projective geometry). *Let (X, cl) be a connected modular geometry.*

1. *If $rank(X) = 1$, then X is a singleton.*
2. *If $rank(X) = 2$, then X is a "projective line:" X has at least 3 points, and the closure of two distinct points is all of X.*
3. *If $rank(X) = 3$, then X is a "projective plane:"*

[18] Also known as a division ring...

[19] A (bounded) lattice is a poset $(P, \leq)$ in which every finite set has an infimum and a supremum. One only needs to check the empty set and sets of size 2, so an equivalent condition is that P is a (bounded) lattice if P has a maximum 1 and a minimum 0, and for every $x, y \in P$ the set $\{x, y\}$ has an infimum $x \wedge y$ and a supremum $x \vee y$. The partial order is determined by algebraic structure $(P, 0, 1, \wedge, \vee)$, and one can write down algebraic identities which characterize lattices. So alternatively, one can view lattices as algebraic structure $(P, 0, 1, \wedge, \vee)$ satisfying some list of identities. For example, boolean algebras are a type of lattice. A *modular lattice* is a lattice satisfying the identity $(x \wedge b) \vee a = (x \vee a) \wedge b$ whenever $a \leq b$. This identity ensures that the Jordan-Hölder theorem works out: if $x = c_0 < c_1 < \cdots < c_n = y$ and $x = c'_0 < c'_1 < \cdots < c'_m = y$, and if there are no z strictly between c_i and c_{i+1} or between c'_i and c'_{i+1}, then $n = m$. If G is a group, the lattice of normal subgroups of G is always a modular lattice. If M is an R-module, then the lattice of R-submodules is always a modular lattice. (This is probably the origin for the term "modular.") It turns out that a pregeometry (X, cl) is modular iff the lattice of closed sets is modular. In fact, modular geometries are equivalent to atomic modular lattices, where a modular lattice $(P, \leq)$ is *atomic* if every element is of the form $a_1 \vee \cdots \vee a_n$ with the a_i atoms.

- *X is a set of points*
- *The rank-1 closed sets in X are a set of "lines"*
- *Any two points lie on exactly one line.*
- *Any two lines intersect in exactly one point.*

4. *If rank$(X) \geq 4$, then X is projective space*[20] *over some division ring.*

It would be nice if every projective plane was a projective plane over a division ring, but this only holds for projective planes satisfying Desargues theorem. For model theory, this isn't a big deal, because most of the pregeometries we care about have infinite rank.

16 The false Zilber trichotomy conjecture

Fact 16.1. *If $K \models ACF$, the pregeometry (K, acl) is* not *modular. In fact, if a, b, x are independent transcendentals over the prime field F and*

$$y = ax + b,$$

and $K_1 = \mathrm{acl}(a, b)$ and $K_2 = \mathrm{acl}(x, y)$, then

$$\begin{aligned} \mathrm{tr.deg}(K_1/F) &= 2 \\ \mathrm{tr.deg}(K_2/F) &= 2 \\ \mathrm{tr.deg}(K_1K_2/F) &= 3 \\ \mathrm{tr.deg}((K_1 \cap K_2)/F) &= 0 \end{aligned}$$

but $3 \neq 2 + 2 - 0$.

At one time, it was hoped that

Conjecture 16.2 (Zilber trichotomy conjecture)**.** *If M is a strongly minimal structure, then either*

- *(M, acl) is a modular pregeometry.*
- *M is bi-interpretable with an algebraically closed field.*

So one hoped that ACF was the "only" complicated strongly minimal structure.
If (M, acl) is a modular pregeometry, it turns out that one of two things happens:

- The geometry (M, acl) is "trivial": the associated pregeometry is a disconnnected union of points, and $\mathrm{acl}(X) = \bigcup_{p \in X} \mathrm{acl}(\{p\})$.
- The geometry (M, acl) is projective space over a division ring.

[20]Possibly infinite dimensional...

These two possibilities, plus the ACF case of Conjecture 16.2, are the "trichotomy."

One can say something about groups and fields in each case:

1. If (M, acl) has a trivial geometry, there are no infinite definable groups or fields.
2. If (M, acl) has a projective geometry, there is a 1-dimensional definable abelian group $(G, +)$.[21] The set G is strongly minimal, and is in finite-to-finite correspondence to M. So M essentially "is" G. There are no definable fields. The structure of definable sets on G^n is constrained by Fact 16.4 below.
3. If M is bi-interpretable with an algebraically closed field K, then K is the only definable field, and the definable groups are algebraic groups over K.

In case 2, it turns out that every connected definable group G^0 is abelian (See Fact 16.4 below). So the Zilber trichotomy conjecture implies

Conjecture 16.3 (Cherlin-Zilber)**.** *If G is a simple non-abelian group interpretable in a strongly minimal theory, then G is an algebraic group over an algebraically closed field. More generally*[22]*, if G is a simple non-abelian group of finite Morley rank, then G is an algebraic group over an algebraically closed field.*

Ultimately, Hrushovski disproved Conjecture 16.2, finding a large bank of counterexamples via a complicated variant of Fraisse limits.

Nevertheless, there are several special situations in which Conjecture 16.2 is known.

- If M is $\aleph_0$-categorical, then M is modular, hence satisfies the Zilber trichotomy conjecture.
- If M is pseudo-finite (elementarily equivalent to an ultraproduct of finite structures), then M is modular, hence satisfies the Zilber trichotomy conjecture.[23]
- If M admits a "Zariski topology"[24], then M satisfies the Zilber trichotomy conjecture: either M interprets an algebraically closed field or M is modular.

This third point—the case where a Zariski topology exists—can be used to verify the trichotomy in DCF_0, the theory of differentially closed fields of characteristic 0. More specifically, if $K \models DCF_0$ and X is a K-definable strongly minimal set, then *either*

[21] The proof uses Hrushovski's *group configuration theorem*. See e.g. Theorem V.4.5 in Pillay's *Geometric Stability Theory*.

[22] See Fact 11.1 and the comments thereafter.

[23] Interestingly, both this and the $\aleph_0$-categorical case use the classification of group actions on strongly minimal sets (Theorem 9.12 in Friday's notes), in combination with Hrushovski's group configuration theorem. See Proposition II.4.15 and V.3.2 in Pillay's *Geometric Stability Theory* for the essential points.

[24] In algebraic geometry, the Zariski topology on K^n (for $K \models ACF$) is the topology in which $V \subseteq K^n$ is closed iff V is the set $V(P_1, \ldots, P_n) = \{\vec{x} \in K^n : P_1(\vec{x}) = \cdots = P_n(\vec{x}) = 0\}$ for some polynomials $P_1, \ldots, P_n$ in $K[X_1, \ldots, X_n]$. This topology is a bit unintuitive, because it fails to be Hausdorff. On the other hand, it is compact and T_1. If M is a structure, an abstract "Zariski topology" on M is a collection of topologies on M^n for each n, satisfying abstract properties chosen by Zilber and Hrushovski. See the book *Zariski geometries* by Zilber for details.

- The acl-pregeometry on X is modular, OR
- X is an algebraic curve over the "field of constants" C

$$C = \{x \in K : \partial x = 0\}.$$

This fact has been used to deduce new results about differential fields and differential algebraic geometry.

In the setting of groups of finite Morley rank, it is worth noting the following fact:

Fact 16.4. *Let G be a definable group of finite Morley rank which is "governed"[25] by finitely many strongly minimal sets $X_1, \ldots, X_n$. Suppose that every X_i has a modular geometry. Then*

- *The connected component G^0 is abelian.*
- *Every definable subset $X \subseteq G^n$ is a finite boolean combination of cosets of definable subgroups of G^n.*

This forces the induced structure on G to be very restricted, basically just an R-module for some ring R.

(See §IV.4 in Pillay's *Geometric Stability Theory* for details.)

The conclusion of Fact 16.4 looks obscure, but turns out to be tightly related to certain problems in diophantine geometry, like the Mordell-Lang conjecture[26] The characteristic p analogue of the Mordell-Lang conjecture was proven by Hrushovski, by

- Using Zariski geometries to prove an appropriate analogue of the Zilber trichotomy in the theory SCF of separably closed fields.
- Using properties of stable groups, including variants of Fact 16.4, to prove the desired result.

For more details, see the MSRI publication *Model Theory, Algebra, and Geometry*, or the book *Model Theory and Algebraic Geometry: an Introduction to E. Hrushovski's proof of the geometric Mordell-Lang Conjecture*, edited by Elisabeth Bouscaren.

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[25] In other words, if $M' \succeq M$ and $G(M')$ is strictly greater than $G(M)$, then $X_i(M') \supsetneq X_i(M)$ for some i.

[26] This says: if A is an abelian variety over an algebraically closed field K of characteristic 0, and Γ is a finitely generated subgroup of $A(K)$, and $V \subseteq A$ is an irreducible subvariety such that $V(K) \cap \Gamma$ is Zariski-dense in V, then V is a coset of an algebraic subgroup. The motivation is that if C is a curve over $\mathbb{Q}$ of genus at least 1, and A is the Jacobian variety of C, then C embeds into A as a subvariety, and the $\mathbb{Q}$-rational points $\Gamma = A(\mathbb{Q})$ form a finitely-generated subgroup of $A(\mathbb{Q}^{alg})$ by the Mordell-Weil theorem. If C had infinitely many rational points, then $C \cap \Gamma$ would be infinite, hence Zariski dense in C. So C would need to be a definable coset. The only 1-dimensional definable subgroups of A are elliptic curves, so C would have genus 1. Thus the Mordell-Lang conjecture implies the Mordell conjecture (also proven by Faltings).

A Stone duality

A *boolean algebra* is a structure $(B, \cap, \cup, 0, 1, \neg)$ satisfying some equational axioms

$$\begin{aligned} x \cup y &= y \cup x \\ x \cup x &= x \\ x \cup (x \cap y) &= y \\ x \cap (y \cup z) &= (x \cap y) \cup (x \cap z) \\ &\dots \end{aligned}$$

More importantly, boolean algebras are characterized as follows:

Fact A.1. *If X is a set, then $(Pow(X), \cap, \cup, 0, 1, \neg)$ is a boolean algebra, where $0 = \emptyset$, $1 = X$, and $\neg S = (X \setminus S)$.*

Moreover, a structure $(B, \cap, \cup, 0, 1, \neg)$ is a Boolean algebra iff it is isomorphic to a subalgebra of $Pow(X)$ for some set X.

So up to isomorphism, a boolean algebra is essentially a non-empty collection B of subsets of a set X, such that B is closed under unions, intersections, and complements.

There is a category *Bool* of boolean algebras, in which the morphisms are the maps $B_1 \to B_2$ preserving all the structure $0, 1, \cap, \cup, \neg$.

Definition A.2. *A* stone space *is a topological space S which is compact and totally disconnected: for any $a, b \in S$ with $a \neq b$, there is a clopen set $U \subseteq S$ such that $a \in U$ and $b \notin U$.*

For example, the cantor set is a stone space, and so is the one-point compactification of $\mathbb{N}$. The interval $[0, 1]$ isn't a stone space, because it's not totally disconnected. The rationals $\mathbb{Q}$ are totally disconnected, but not compact, so they aren't a stone space either.

There is a category *Stone* of stone spaces, with morphisms being the continuous maps.

If S is a stone space, there is a boolean algebra $B(S)$ of clopen subsets of S. A continuous map $f : S_1 \to S_2$ of stone spaces induces a pullback morphism of boolean algebras

$$\begin{aligned} B(S_2) &\to B(S_1) \\ U &\mapsto f^{-1}(U). \end{aligned}$$

So B is a contravariant functor from *Stone* to *Bool*.

Theorem A.3 (Stone duality)**.** *The functor B is an anti-equivalence of categories from Stone to Bool, i.e., B is an equivalence of categories from $Stone^{op}$ to Bool.*

So there is a contravariant functor $S(-)$ from boolean algebras back to stone spaces, and natural isomorphisms

$$\begin{aligned} B(S(X)) &\cong X \qquad \text{for } X \in \textit{Bool} \\ S(B(X)) &\cong X \qquad \text{for } X \in \textit{Stone}. \end{aligned}$$

What is the stone space $S(X)$ associated to a boolean algebra X? From the anti-equivalence, we can at least read off the underlying set:

$$|S(X)| \cong \mathrm{Hom}_{Stone}(*, S(X)) \cong \mathrm{Hom}_{Bool}(X, B(*)) \cong \mathrm{Hom}_{Bool}(X, \{0,1\}),$$

where $*$ is the one-point set and $B(*) = \{0,1\}$ is the two-element boolean algebra. So the points of $S(X)$ correspond exactly to Boolean-algebra homomorphisms from X to $\{0,1\}$. Giving a function from X to $\{0,1\}$ is equivalent to giving a subset $F \subseteq X$, and the resulting function is a homomorphism iff F is an ultrafilter on B. So

The stone space $S(X)$ associated to X is the set of ultrafilters on X.

Stone duality preserves several properties of morphisms. For example, a morphism of boolean algebras

$$f : B_1 \to B_2$$

is injective iff and only if the dual morphism

$$f^* : S_2 \to S_1$$

of stone spaces is *surjective*.

B Type spaces and quantifier elimination

If M is a structure and $n \geq 1$, then there is a boolean algebra B_n consisting of the definable subsets of M^n.

Theorem B.1. *Let $A \subseteq M$ be a subset.*

1. *The A-definable subsets of M^n form a boolean subalgebra $B_n(A) \leq B_n$.*
2. *An ultrafilter on $B_n(A)$ is equivalent to a (complete) n-type $p(x_1, \ldots, x_n)$ over A.*
3. *The stone space dual to $B_n(A)$ is the space $S_n(A)$ of n-types over A. So $S_n(A)$ admits a natural stone space topology.*
4. *The boolean algebra of clopen sets in $S_n(A)$ is isomorphic to $B_n(A)$.*

One can play the same game with quantifier-free definable sets and quantifier-free types. Let $B_n^{qf}(A)$ denote the boolean algebra of sets quantifier-free definable over A, and let $S_n^{qf}(A)$ denote the space of quantifier-free n-types over A. Then $S_n^{qf}(A)$ is naturally a stone space dual to $B_n^{qf}(A)$.

There is an inclusion of boolean algebras $B_n^{qf}(\emptyset) \hookrightarrow B_n(\emptyset)$. By Stone duality, one gets the following equivalence:

Theorem B.2. *Let M be a structure. The following are equivalent:*

1. *The theory $Th(M)$ has quantifier elimination.*

2. *For every n, the inclusion*
$$B_n^{qf}(\emptyset) \hookrightarrow B_n(\emptyset)$$
is onto, hence an isomorphism.

3. *For every n, the continuous surjection*
$$S_n(\emptyset) \twoheadrightarrow S_n^{qf}(\emptyset)$$
is injective, hence an isomorphism.[27]

4. *For every $M' \succeq M$ and every $\vec{a}, \vec{b} \in (M')^n$,*
$$\mathrm{qftp}(\vec{a}/\emptyset) = \mathrm{qftp}(\vec{b}/\emptyset) \implies \mathrm{tp}(\vec{a}/\emptyset) = \mathrm{tp}(\vec{b}/\emptyset).$$

The equivalences $1 \iff 2$ and $3 \iff 4$ are by definition, and the equivalence $2 \iff 3$ is by stone duality.

Theorem B.2 was essentially the criterion we used to prove quantifier elimination for ACF, with all the complicated compactness arguments condensed into Stone duality. More precisely, we used the following criterion to show that an individual set is quantifier-free definable:

Theorem B.3. *Let X be an $\emptyset$-definable subset of M^n. Let X^* be the corresponding clopen set in $S_n(\emptyset)$. Then X is quantifier-free definable over $\emptyset$ iff X^* is the preimage of some set under the surjection*
$$S_n(\emptyset) \twoheadrightarrow S_n^{qf}(\emptyset).$$
In other words, X is quantifier-free iff for every $M' \succeq M$ and every $\vec{a}, \vec{b} \in (M')^n$,
$$\mathrm{qftp}(\vec{a}/\emptyset) = \mathrm{qftp}(\vec{b}/\emptyset) \implies (\vec{a} \in X \iff \vec{b} \in X).$$

(Compare with Lemma 6.5 in Monday's notes.) This theorem holds because of the following exercise in topology:

Exercise B.4. *Let $f : S_1 \twoheadrightarrow S_2$ be a surjective map of stone spaces, and $X \subseteq S_1$ be clopen. If $X = f^{-1}(Y)$ for some $Y \subseteq S_2$, then Y is clopen.*

[27] If $f : S_1 \to S_2$ is a continuous bijection between stone spaces, then f is a homeomorphism (an isomorphism). This doesn't hold for general topological spaces, but does hold for compact Hausdorff spaces.

C Cantor-Bendixson rank

If X is a closed set in a topological space, the *derived set* X' is the set of $a \in X$ such that a is in the closure of $X \setminus \{a\}$. So X' is X minus the isolated points in X. For example, if $X = \{1, 1/2, 1/3, 1/4, \ldots, 0\} \subseteq \mathbb{R}$, then $X' = \{0\}$, and $(X')' = \emptyset$. A *perfect set* is a non-empty closed set X without isolated points, so $X = X'$.

Fix a stone space S. Define a sequence of closed sets

$$S = S^{(0)} \supseteq S^{(1)} \supseteq \cdots S^{(\omega)} \supseteq \cdots$$

by transfinite induction as follows:

- $S^{(0)} = S$
- $S^{(\alpha+1)}$ is the derived set $(S^{(\alpha)})'$.
- If α is a limit ordinal, then
$$S^{(\alpha)} = \bigcap_{\beta<\alpha} S^{(\beta)}$$

The sequence must eventually stabilize at some closed set $S^{(\infty)} \subseteq S$. The set $S^{(\infty)}$ is either empty or a perfect set.

Theorem C.1. *If $V \subseteq S$ is perfect, there is a tree $\{U_a\}_{a\in\{0,1\}^{<\omega}}$ of clopen subsets of S such that*

- $U_\emptyset = S$
- *For any $a \in \{0,1\}^{<\omega}$, the set U_a is the disjoint union of U_{a0} and U_{a1}.*
- *For any $a \in \{0,1\}^{<\omega}$, the set U_a intersects V.*

For every $a \in \{0,1\}^\omega$, the intersection

$$X \cap \bigcap_{n=1}^{\infty} U_{a\restriction n}$$

is non-empty by compactness, and so $|X| \geq 2^{\aleph_0}$.

Proof sketch. One defines U_a by induction on $|a|$, using perfection of V to find two distinct points in $V \cap U_a$, and total disconnectedness of S to split U_a into two clopen sets U_{a0} and U_{a1}. □

To every point $p \in S$, we can associate a "Cantor-Bendixson rank" $CB(p)$, which is the maximum α such that $p \in S^{(\alpha)}$, or ∞ if $p \in S^{(\infty)}$.

Very concretely, $CB(p)$ is characterized by the fact that $CB(p) \geq \alpha + 1$ iff for every neighborhood $U \ni p$, there are infinitely many points $p' \in U$ with $CB(p') \geq \alpha$.

If $X \subseteq S$ is closed, then the set

$$\{CB(p) : p \in X\}$$

has a maximum, essentially by compactness. We denote this maximum by $CB(X)$, the Cantor-Bendixson rank of X. If $CB(X) < \infty$, then there are only finitely many $p \in X$ such that $CB(p) = CB(X)$. (Otherwise, these points would accumulate at some $q \in X$, and then $CB(q) \geq CB(X) + 1$, a contradiction.) The size of the finite set

$$\{p \in X : CB(p) = CB(X)\}$$

is the "Cantor-Bendixson degree" of X.

Using Cantor-Bendixson ranks and degrees, one can prove

Theorem C.2. *Let S be a stone space. Then the following are equivalent:*

- $S^{(\infty)} \neq \emptyset$.
- *S contains a perfect closed set.*
- *There is a tree $\{U_a\}_{a \in \{0,1\}^{<\omega}}$ of non-empty clopen sets, such that for every $a \in \{0,1\}^{<\omega}$, the set U_a is the disjoint union of U_{a0} and U_{a1}.*

One proves $1 \implies 2 \implies 3$. If 3 holds and 1 fails, one can find a path through the tree on which $(CB(U_a), deg(U_a))$ is strictly decreasing, contradicting DCC on ordinals.

If there is a basis of size κ for the topology on S, then $|S \setminus S^{(\infty)}| \leq \kappa$. (Each basic open is responsible for removing at most one point from $S^{(\infty)}$.) When S is dual to a countable boolean algebra, then S is second countable. Using Theorem C.1, one gets

Theorem C.3. *Let S be dual to countable boolean algebra B. Then the following are equivalent:*

- $S^{(\infty)} = \emptyset$
- $|S| \leq \aleph_0$
- $|S| < 2^{\aleph_0}$.
- *In B there is a tree $\{x_a\}_{a \in \{0,1\}^{<\omega}}$ where*

$$\begin{aligned} x_a &> 0 \\ x_{a0} \cap x_{a1} &= 0 \\ x_{a0} \cup x_{a1} &= x_a \end{aligned}$$

for all $a \in \{0,1\}^{<\omega}$.

The equivalence of 1, 4 holds without the assumption that B is countable. In particular, $S_n(B)$ contains a perfect set iff there is a countable subalgebra $B' \leq B$ such that $S_n(B')$ is uncountable.

D CB-rank and Morley rank

If S is a stone space, $X \subseteq S$ is clopen, and α is an ordinal, then one can verify that the following are equivalent:

- $CB(X) \geq \alpha + 1$
- For every $d < \omega$, there exist disjoint clopens $X_1, \ldots, X_d \subseteq X$ with $CB(X_i) \geq \alpha$.
- There exist disjoint clopens $X_1, X_2, X_3, \ldots \subseteq X$ with $CB(X_i) \geq \alpha$.

If $\mathbb{M}$ is a monster model and S is the space $S_n(\mathbb{M})$ of n-types over $\mathbb{M}$, then Cantor-Bendixson rank and degree in S are therefore exactly Morley rank and Morley degree. The structure $\mathbb{M}$ is totally transcendental iff $S^{(\infty)} = \emptyset$. If the language is countable, then the boolean algebra $B_n(\mathbb{M})$ is a directed union of countable boolean algebras of the form $B_n(M_0)$ where M_0 ranges over countable models $M_0 \preceq \mathbb{M}$. Thus $\mathbb{M}$ is totally transcendental iff $|S_n(M_0)| \leq \aleph_0$ for every countable $M_0 \preceq \mathbb{M}$. This explains the equivalence

$$T \text{ is } \aleph_0\text{-stable} \iff T \text{ is totally transcendental}$$

mentioned in Fact 5.4.