Strongly minimal theories: Wednesday notes

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August 14, 2019

1 Rank of tuples

Let $\mathbb{M}$ be a strongly minimal monster model.

Recall from last time

Lemma 1.1. *If a, b are singletons, $C \subseteq \mathbb{M}$ is small, and $a, b \notin \operatorname{acl}(C)$, then $\operatorname{tp}(a/C) = \operatorname{tp}(b/C)$.*

Lemma 1.2. *If C is small, there exists $a \in \mathbb{M}$ such that $a \notin \operatorname{acl}(C)$.*

Say that a tuple $\vec{a}$ is *C-independent* if it is independent over C in the sense of the pregeometry $(\mathbb{M}, \operatorname{acl}(-))$:

$$a_{i+1} \notin \operatorname{acl}(C \cup \{a_1, \ldots, a_i\})$$

for $i < |\vec{a}|$. Note this is permutation invariant, in spite of appearances to the contrary.

If $\sigma \in \operatorname{Aut}(\mathbb{M}/C)$, then $\vec{a}$ is C-independent iff $\sigma(\vec{a})$ is C-independent, by symmetry. Therefore, whether or not $\vec{a}$ is C-independent depends only on $\operatorname{tp}(\vec{a}/C)$.

Lemma 1.3. *For any small $C \subseteq \mathbb{M}$, there is a unique n-type p over C such that $\vec{a} \models p$ if and only if $\vec{a}$ is C-independent.*

Proof. Existence: inductively choose $a_1, a_2, \ldots, a_n$ such that $a_i \notin \operatorname{acl}(Ca_1, \ldots, a_{i-1})$ for each i.

Uniqueness: if $\vec{a}$ and $\vec{b}$ are both independent over C, we claim there is $\sigma \in \operatorname{Aut}(\mathbb{M}/C)$ such that $\sigma(\vec{a}) = \vec{b}$. We prove this by induction on n. By induction, there is $\sigma_0 \in \operatorname{Aut}(\mathbb{M}/C)$ such that

$$\sigma_0(a_1, \ldots, a_{n-1}) = (b_1, \ldots, b_{n-1}).$$

Let $a'_i = \sigma_0(a_i)$, so $a'_i = b_i$ for $i < n$. Then $\vec{a}'$ is independent over C, so $a'_n \notin \operatorname{acl}(Ca'_1, \ldots, a'_{n-1}) = \operatorname{acl}(Cb_1 \ldots b_{n-1})$. Therefore

$$a'_n \equiv_{Cb_1 \cdots b_{n-1}} b_n,$$

and there is $\sigma_1 \in \operatorname{Aut}(\mathbb{M}/Cb_1 \cdots b_{n-1})$ such that $\sigma_1(a'_n) = b_n$. Then

$$\sigma_1(\sigma_0(a_i)) = \sigma_1(b_i) = b_i \qquad \forall i < n$$
$$\sigma_1(\sigma_0(a_n)) = \sigma_1(a'_n) = b_n$$

So $\sigma_1 \circ \sigma_0 \in \operatorname{Aut}(\mathbb{M}/C)$ carries $\vec{a}$ to $\vec{b}$. $\square$

If $\vec{a}$ is a finite tuple and C is a set, the *rank* $R(\vec{a}/C)$ is the rank of $\vec{a}$ over C, i.e., the size of a maximal C-independent subtuple of $\vec{a}$.

Remark 1.4. $0 \leq R(\vec{a}/C) \leq |\vec{a}|$

Proposition 1.5. *If $\vec{a} \equiv_C \vec{b}$, then $R(\vec{a}/C) = R(\vec{b}/C)$.*

Proof. By homogeneity, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma(\vec{a}) = \vec{b}$. Then $R(\vec{a}/C) = R(\vec{b}/C)$ by symmetry. □

Definition 1.6. *Two tuples $\vec{a}, \vec{b}$ are* interalgebraic over C *if $\vec{a} \in \mathrm{acl}(C\vec{b})$ and $\vec{b} \in \mathrm{acl}(C\vec{a})$. Equivalently, $\mathrm{acl}(C\vec{a}) = \mathrm{acl}(C\vec{b})$.*

Proposition 1.7. *If $\vec{a}, \vec{b}$ are interalgebraic over C, then $R(\vec{a}/C) = R(\vec{b}/C)$.*

Proposition 1.8. *If $C \subseteq C'$, then $R(\vec{a}/C) \geq R(\vec{a}/C')$.*

Proposition 1.9 (Lascar equality)**.** $R(\vec{a}\vec{b}/C) = R(\vec{a}/\vec{b}C) + R(\vec{b}/C)$.

We proved all these facts last time, in greater generality. We also need the following variant of Proposition 1.7

Proposition 1.10. *If $\vec{a} \subseteq \mathrm{acl}(C\vec{b})$, then $R(\vec{a}/C) \leq R(\vec{b}/C)$.*

Proof. More generally in any pregeometry, if $A \subseteq cl(BC)$ then $R(A/C) \leq R(B/C)$. Indeed, let $A_0 \subseteq A$ be a basis of A over C. Then A_0 is a C-independent subset of AB, so A_0 can be extended to a basis of AB over C. Thus

$$rank(A/C) = |A_0| \leq rank(AB/C) = rank(B/C),$$

where the final equality holds because $cl(ABC) = cl(BC)$. □

The following is an easy exercise using pregeometries:

Proposition 1.11. *$R(\vec{a}/C) = 0$ iff $\vec{a}$ is a tuple from $\mathrm{acl}(C)$.*

The following fact is specific to the model-theoretic setting:

Lemma 1.12 (Extension)**.** *If $C \subseteq C'$ and $\vec{a}$ is a tuple, there is $\vec{a}'$ such that*

$$\vec{a}' \equiv_C \vec{a}$$
$$R(\vec{a}'/C') = R(\vec{a}/C)$$

Proof. Let $\vec{b}$ be a basis of $\vec{a}$ over C. Let $\vec{b}'$ be a C'-independent tuple of length $|\vec{b}|$. Then $\vec{b}'$ and $\vec{b}$ are both C-independent tuples of length $|\vec{b}|$, so $\vec{b}' \equiv_C \vec{b}$ and there is $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma(\vec{b}) = \vec{b}'$. Let $\vec{a}' = \sigma(\vec{a})$. Then $\vec{a}' \equiv_C \vec{a}$ and

$$R(\vec{a}/C) = R(\vec{a}'/C) \geq R(\vec{a}'/C') \geq R(\vec{b}'/C') = |\vec{b}'| = |\vec{b}| = R(\vec{a}/C).$$

□

2 Dimension of sets

Definition 2.1. *Let $X \subseteq \mathbb{M}^n$ be type-definable over small C. The* dimension of X[1] $\dim(X)$ *is the maximum of $R(\vec{a}/C)$ for $\vec{a} \in X$, or $-\infty$ if X is empty.*

Proposition 2.2. *This depends only on X, not on C.*

Proof. Suppose X is type-definable over both C and C'. For any $\vec{a} \in X$,

$$R(\vec{a}/C) \geq R(\vec{a}/CC')$$

so

$$\max_{\vec{a} \in X} R(\vec{a}/C) \geq \max_{\vec{a} \in X} R(\vec{a}/CC').$$

On the other hand, take $\vec{a}_0$ with $R(\vec{a}_0/C) = \max_{\vec{a} \in X} R(\vec{a}/C)$. Take $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $R(\sigma(\vec{a}_0)/CC') = R(\vec{a}_0/C)$. Note σ fixes X setwise because X is type-definable over C. Therefore

$$\max_{\vec{a} \in X} R(\vec{a}/CC') \geq R(\sigma(\vec{a}_0)/CC') = R(\vec{a}_0/C) = \max_{\vec{a} \in X} R(\vec{a}/C).$$

We have shown

$$\max_{\vec{a} \in X} R(\vec{a}/C) = \max_{\vec{a} \in X} R(\vec{a}/CC'),$$

and a similar argument shows

$$\max_{\vec{a} \in X} R(\vec{a}/C') = \max_{\vec{a} \in X} R(\vec{a}/CC').$$

□

Lemma 2.3. *Let $f : X \to Y$ be a definable map between two definable sets. Suppose every fiber $f^{-1}(\vec{y})$ has dimension k. Then $\dim(X) = k + \dim(Y)$.*

Proof. Let C be a set over which X, Y, f are defined. Note that any $\vec{a} \in X$ is interalgebraic over C with $\vec{a} f(\vec{a})$, and so

$$R(\vec{a}/C) = R(\vec{a}/Cf(\vec{a})) + R(f(\vec{a})/C) \leq k + \dim(Y),$$

because $\vec{a}$ is an element of the $Cf(\vec{a})$-definable fiber $f^{-1}(\vec{a})$. This shows

$$\dim(X) \leq k + \dim(Y).$$

For the reverse inequality, choose $\vec{b} \in Y$ such that $R(\vec{b}/C) = \dim(Y)$. Then the fiber $f^{-1}(\vec{b})$ is $C\vec{b}$-definable of rank k, so there is $\vec{a} \in f^{-1}(\vec{b})$ with $R(\vec{a}/C\vec{b}) = k$. Then $\vec{a}$ is interalgebraic with $\vec{a} f(\vec{a}) = \vec{a}\vec{b}$ (over C), so

$$R(\vec{a}/C) = R(\vec{a}\vec{b}/C) = R(\vec{a}/\vec{b}C) + R(\vec{b}/C) = k + \dim(Y),$$

showing $\dim(X) \geq k + \dim(Y)$. □

[1] Also called the *rank* of X.

Theorem 2.4. *Let X, Y be definable sets.*

1. $\dim(X \cup Y) = \max(\dim(X), \dim(Y))$, *when the union makes sense.*
2. *If $f : X \to Y$ is a definable surjection, then* $\dim(X) \geq \dim(Y)$.
3. *If $f : X \to Y$ is a definable injection, or more generally a map with finite fibers, then* $\dim(X) \leq \dim(Y)$.
4. *If $f : X \to Y$ is a definable bijection, then* $\dim(X) = \dim(Y)$.
5. $\dim(X \times Y) = \dim(X) + \dim(Y)$.
6. $\dim(X) = 0$ *if and only if X is finite.*
7. $\dim(\mathbb{M}^n) = n$.

Proof. Take a small set C over which X, Y (and f) are defined.

1. Clear from the definition
2. For any $\vec{b} \in Y$ there is $\vec{a} \in X$ such that $f(\vec{a}) = \vec{b}$, and so
$$\dim(X) \geq R(\vec{a}/C) = R(\vec{a}f(\vec{a})/C) \geq R(\vec{b}/C),$$
implying $\dim(X) \geq \dim(Y)$.
3. Note that for any $\vec{a} \in X$, the elements $\vec{a}$ and $f(\vec{a})$ are interalgebraic over C. Then
$$R(\vec{a}/C) = R(f(\vec{a})/C) \leq \dim(Y),$$
and so $\dim(X) \leq \dim(Y)$.
4. Follows from 2 and 3.
5. Let $k = \dim(X)$. Let $\pi : X \times Y \to Y$ be the projection $\pi(x, y) = y$. Then every fiber $\pi^{-1}(b)$ is in definable bijection with X, thus has rank k. By the Lemma,
$$\dim(X \times Y) = k + \dim(Y) = \dim(X) + \dim(Y).$$
6. Suppose X is finite. Then $\vec{a} \in \operatorname{acl}(C)$ for every $\vec{a} \in X$. Thus $\dim(X) = 0$. Conversely, suppose $\dim(X) = 0$. Then $\vec{a} \in \operatorname{acl}(C)$ for every $\vec{a} \in X$. It follows that X is small— $|X| < \kappa$ where $\mathbb{M}$ is κ-saturated. So X is covered by a small number of definable sets (namely singletons). By κ-compactness, there is a finite subcover. Therefore X is finite.
7. By (5), we reduce to the case $n = 0$ or $n = 1$. For $n = 1$, if $a \in \mathbb{M}^1$, then a is a singleton, so $R(a/C) \leq 1$. Thus $\dim(\mathbb{M}) \leq 1$. On the other hand, $\mathbb{M}$ is infinite so $\dim(\mathbb{M}) = 1$ by (6). If $n = 0$, then $\mathbb{M}^0$ is finite, so $\dim(\mathbb{M}^0) = 0$ by (6).

□

Theorem 2.5. *Let $\{X_b\}_{b \in Y}$ be a definable family of subsets of $\mathbb{M}^n$. For each $k \in \mathbb{Z}$, the set*

$$Y_k := \{b \in Y : \dim(X_b) = k\}$$

is definable.

We prove this in the appendix to today's notes.

3 Morley degree

Definition 3.1. *Let X be a non-empty definable set. The* (Morley) degree *of X, written* $\deg(X)$*, is the maximum d such that there are pairwise disjoint definable $X_1, \ldots, X_d \subseteq X$ with* $\dim(X_i) = \dim(X)$*.*

So the degree is always at least 1. We will see soon that $\deg(X) < \infty$.

Example 3.2. *If X is finite,* $\deg(X) = |X|$.

Lemma 3.3. $\deg(\mathbb{M}^n) = 1$.

Proof. Otherwise, take definable $X_1, X_2 \subseteq \mathbb{M}^n$ of rank n, with $X_1 \cap X_2 = \emptyset$. Take small $C \subseteq \mathbb{M}$ over which X_1, X_2 are defined. Take $\vec{a}_i \in X_i$ with $R(\vec{a}_i/C) = n$. Then $\vec{a}_1, \vec{a}_2$ are both C-independent. By an earlier lemma,

$$\vec{a}_1 \equiv_C \vec{a}_2,$$

contradicting the fact that $\vec{a}_1 \in X_1$ and $\vec{a}_2 \notin X_1$. □

Proposition 3.4. *Suppose $X \cap Y = \emptyset$.*

- *If* $\dim(X) > \dim(Y)$*, then* $\deg(X \cup Y) = \deg(X)$.
- *If* $\dim(X) = \dim(Y)$*, then* $\deg(X \cup Y) = \deg(X) + \deg(Y)$.

Proof. In both cases, the inequalities $\deg(X \cup Y) \geq \cdots$ are clear. Conversely, suppose $Z_1, \ldots, Z_d \subseteq X \cup Y$ are pairwise disjoint sets of rank $n := \dim(X)$. For each i, at least one of $Z_i \cup X$ or $Z_i \cup Y$ has full rank n. So in the following list of pairwise disjoint subsets of $X \cup Z$, at least d of the sets have full rank n:

$$Z_1 \cap X, \ldots, Z_d \cap X, Z_1 \cap Y, \ldots, Z_d \cap Y.$$

Thus $\deg(X) + \deg(Y) \geq d$. Furthermore, if $\dim(Y) < n$, then all of the full rank sets are $Z_i \cap D$, showing $\deg(X) \geq d$. □

Corollary 3.5. *If $\deg(X) = d < \infty$ and $X_1, \ldots, X_d$ are pairwise-disjoint definable subsets of X with* $\dim(X_i) = \dim(X)$*, then* $\deg(X_i) = 1$ *for each i.*

Lemma 3.6. *Suppose* $\deg(X) = 1$. *Let* $Y_1, \ldots, Y_k$ *be definable subsets with* $\dim(Y_i) = \dim(X)$. *Then*

$$\dim\left(\bigcap_{i=1}^{k} Y_i\right) = \dim(X).$$

Proof. It suffices to consider $k = 2$. If $\dim(Y_1 \cap Y_2) < \dim(X)$, then

$$\dim(X) = \dim(Y_1) = \max(\dim(Y_1 \cap Y_2), \dim(Y_1 \setminus Y_2))$$

so $\dim(X) = \dim(Y_1) = \dim(Y_1 \setminus Y_2)$. The sets $Y_1 \setminus Y_2$ and Y_2 are disjoint, so $\deg(X) \geq 2$. □

Theorem 3.7. *If* X *is definable* $\deg(X) < \infty$.

(We prove this in the appendix.)

Proposition. *Unlike rank, degree needn't be definable in families.*

4 DCC and connected components

Lemma 4.1. *Let* G *be a definable group and* $H < G$ *be a definable proper subgroup. Then* $(\dim(G), deg(G)) > (\dim(H), deg(H))$ *(with respect to lexicographic order on* $\omega \times \omega$*).*

Proof. As $H \subseteq G$ we have $\dim(G) \leq \dim(H)$. So we may assume $\dim(G) = \dim(H)$. Take $a \in G \setminus H$. Then $H \cap (a \cdot H) = \emptyset$. Note $\deg(a \cdot H) = \deg(H)$.[2] Thus

$$\deg(G) \geq \deg(H) + \deg(a \cdot H) = 2\deg(H) > \deg(H).$$

□

Theorem 4.2. *Let* G *be a definable group, and let* $G_1 \geq G_2 \geq \cdots$ *be a descending chain of definable subgroups. Then* $G_i = G_{i+1} = G_{i+2} = \cdots$ *for some* i.

Proof. $\omega \times \omega$ is well-ordered with respect to lexicographic ordering. □

Corollary 4.3. *If* G *is a definable group, there is a minimal definable subgroup* G^0 *such that* G/G^0 *is finite.*

Remark 4.4.

1. *If* G *is* A*-definable, then so is* G^0*, because it is definable and* A*-invariant, by symmetry.*

2. *If* $f : G \to G$ *is a definable automorphism, then* $f(G^0) = G^0$. *Considering inner automorphisms, we see* $G^0 \lhd G$ *(*G^0 *is a normal subgroup of* G*).*

3. $\dim(G^0) = \dim(G)$*, because* G *is a finite union of cosets of* G^0.

[2] Note that if $f : X \to Y$ is a definable bijection, then $\deg(X) = \deg(Y)$.

Theorem 4.5. *If $G = G^0$, then $deg(G) = 1$.*

Proof. Let $n = \dim(G)$ and $d = \deg(G)$. Let $\mathcal{F}$ be the collection of definable subsets $X \subseteq G$ with $\dim(X) = n$ and $\deg(X) = 1$. For $X, Y \in \mathcal{F}$, let $\sim$ be the relation

$$X \sim Y \iff \dim(X \cap Y) = n.$$

This is an equivalence relation. Fix $X_1, \ldots, X_d$ pairwise disjoint definable subsets of rank n. Then $X_i \in \mathcal{F}$ and the X_i are representatives of the equivalence classes.[3] In particular, $|\mathcal{F}/\sim| = d$.

If $X, Y \in \mathcal{F}$ and $a \in G$, then $a \cdot X, a \cdot Y \in \mathcal{F}$ and

$$X \sim Y \implies a \cdot X \sim a \cdot Y.$$

Therefore G acts on $\mathcal{F}/\sim$ by left-translation. Therefore, for any $X \in \mathcal{F}$ the stabilizer

$$Stab(X) = \{a \in G : a \cdot X \sim X\}$$

is a finite-index subgroup of G. $Stab(X)$ is also definable, because $\dim(X \cap a \cdot X)$ varies definably with a. As $G = G^0$, we see $Stab(X) = G$. We have shown

$$X \sim a \cdot X \text{ for all } X \in \mathcal{F} \text{ and } a \in G.$$

Similarly,

$$X \sim X \cdot a \text{ for all } X \in \mathcal{F} \text{ and } a \in G.$$

Now suppose $d > 1$. Take $X_1, \ldots, X_d$ as above. Take small C defining the X_i. Take

$$(a, b) \in X_1 \times X_2$$

with $R(ab/C) = \dim(X_1 \times X_2) = \dim(X_1) + \dim(X_2) = 2n$. We claim $a \cdot b \in X_1$. Otherwise, $a \cdot b \in (X_1 \cdot b) \setminus X_1$. Now

$$\dim((X_1 \cdot b) \cap X_1) = n \implies \dim((X_1 \cdot b) \setminus X_1) < n$$

because $\deg(X_1 \cdot b) = 1$. The set $(X_1 \cdot b) \setminus X_1$ is bC-definable, so

$$R(a/bC) = R(a \cdot b/bC) \leq \dim((X_1 \cdot b) \setminus X_1) < n.$$

Then

$$2n = R((a, b)/C) = R(a/bC) + R(b/C) < n + R(b/C) \leq 2n,$$

a contradiction. Thus $a \cdot b \in X_1$. A similar argument shows $a \cdot b \in X_2$, contradicting $X_1 \cap X_2 = \emptyset$. $\square$

Corollary 4.6. *$deg(G) = |G/G^0|$ for any definable group G.*

[3] By choice of the X_i, we have $X_i \not\sim X_j$ for $i \neq j$, as $X_i \cap X_j$ is empty, not of dimension n. Also, given any $Y \in \mathcal{F}$, there must be some i such that $Y \cap X_i$ has dimension n. Otherwise, $Y' := Y \setminus \bigcup_{i=1}^{d} X_i$ would have full dimension n, and the collection $\{X_1, \ldots, X_d, Y'\}$ would show $\deg(G) \geq d + 1$, a contradiction.

5 Definable fields

Lemma 5.1. *Let K be a definable infinite field in a strongly minimal theory. Then deg*$(K) =$ 1.

Proof. Let G be the additive group in K. Then G^0 is an ideal, so it is 0 or K. The index of G^0 in G is finite, so $G^0 = K$, and $\deg(K) = 1$. $\square$

Lemma 5.2. *Let K be a definable infinite field in a strongly minimal theory. Then the multiplicative group $K^\times = (K \setminus \{0\}, \cdot)$ is connected:*

$$(K^\times)^0 = K^\times.$$

Proof. It suffices to show $\deg(K \setminus \{0\}) = 1$. But because K is infinite,

$$\deg(K) = \deg(K \setminus \{0\}).$$

$\square$

Lemma 5.3. *Let K be a definable infinite field in a strongly minimal theory. Then for every n, the nth power map $K \to K$ is surjective.*

Proof. Let G be the multiplicative group of K, and H be the subgroup of nth powers. Then $G \to H$ has finite fibers, so $\dim(G) = \dim(H)$, and we see that H has finite index in G.[4] As $G = G^0$, we see $H = G$. $\square$

Next time, we will use this fact as part of Macintyre's theorem that definable infinite fields (in strongly minimal theories) are algebraically closed.

6 Exercises

Work in a monster model $\mathbb{M}$ of a strongly minimal theory.

Exercise 6.1. *Let G be a definable group with a definable group action on X. Suppose the action is transitive. For $p \in X$, let* $\mathrm{Stab}(p)$ *denote the stabilizer $\{g \in G : g \cdot p = p\}$. Show*

$$\dim(X) = \dim(G) - \dim(\mathrm{Stab}(p)).$$

Exercise 6.2. *Let $H \leq G$ be definable groups. If $\dim H = \dim G$, show $|G/H| < \infty$.*

Exercise 6.3. *In $\mathbb{C}$, show*

- *$\{(x, y) : xy = 1\}$ has dimension 1.*
- *$\{(t, t^2, t^3) : t \in \mathbb{C}\}$ has dimension 1.*

[4]See Exercise 6.2 below.

- $\{(x, xy) : x, y \in \mathbb{C}\}$ *has dimension 2.*

Exercise 6.4 (Harder)**.** *Let $\mathbb{M}$ be a monster model of some theory,* not necessarily strongly minimal. *Suppose* $\mathrm{acl}(-)$ *satisfies Steinitz exchange, so* $(\mathbb{M}, \mathrm{acl}(-))$ *is a pregeometry. Define $R(\vec{a}/C)$ as in the strongly minimal case. Show that the Extension Lemma 1.12 holds: given small $C \subseteq C'$ and a finite tuple $\vec{a} \in \mathbb{M}^n$, there is $\vec{a}' \equiv_C \vec{a}$ such that $R(\vec{a}'/C') = R(\vec{a}/C)$.*

Hint: first consider the case where $\vec{a}$ is C-independent. Proceed by induction on $|\vec{a}|$.

Remark 6.5. *If $\mathbb{M}$ satisfies the property of Exercise 6.4, one can mimic the arguments of §2 and prove that there is a good dimension theory. Many mathematical structures of interest have this property but fail to be strongly minimal. For example, the field $\mathbb{R}$ of real numbers has this property, as do the fields $\mathbb{Q}_p$ of p-adic numbers and the rings $\mathbb{R}[[t]]$ and $\mathbb{C}[[t]]$ of formal power series.*

7 Appendix: definability of rank, finiteness of degree

Lemma 7.1. *Let $\phi(\vec{x}; \vec{y})$ be a formula with $|\vec{x}| = n$. For any $\vec{b}$, the following are equivalent:*

- $\phi(\mathbb{M}; \vec{b})$ *has full rank n.*
- $\exists^\infty x_1 \cdots \exists^\infty x_n : \phi(x_1, \ldots, x_n, \vec{b})$.

Proof. By induction on n, the $n = 1$ case being known (Theorem 2.4.6). Consider the formulas

$$\psi(x_1, \ldots, x_{n-1}; \vec{y}) = \exists^\infty x_n : \phi(x_1, \ldots, x_{n-1}, x_n; \vec{y})$$
$$\chi(x_1, \ldots, x_{n-1}; \vec{y}) = \exists x_n : \phi(x_1, \ldots, x_{n-1}, x_n; \vec{y})$$

For any $\vec{b}$, the fibers of the map

$$\phi(\mathbb{M}; \vec{b}) \to \mathbb{M}^{n-1}$$
$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$$

over $\vec{a} = (a_1, \ldots, a_{n-1})$ have rank

1. 1 iff $\vec{a} \in \psi(M; \vec{b})$
2. 0 iff $\vec{a} \in \chi(M; \vec{b}) \setminus \psi(M; \vec{b})$
3. $-\infty$ iff $\vec{a} \notin \chi(M; \vec{b})$.

Thus $\phi(M; \vec{b})$ has rank n if and only if $\psi(M; \vec{b})$ has rank $n + 1$. By induction, this completes the proof. $\square$

Write $X \rightsquigarrow Y$ if there is a definable surjection $X \to Y$ with finite fibers. Note that if $X \rightsquigarrow Y$ then $\dim(X) = \dim(Y)$ by Theorem 2.4.2-3.

Lemma 7.2. *For any definable set X, there are definable subsets $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ such that $X_i \rightsquigarrow Y_i$ and Y_i is a rank k_i-subset of $\mathbb{M}^{k_i}$ for some k_i.*

Proof. Take a small set C defining X. Say that a C-definable subset $X' \subseteq X$ is "good" if there is a C-definable function $f : X' \to M^k$ with finite fibers, where $k = \dim(X')$. By compactness (saturation), it suffices to show that every $\vec{a} \in X$ is in a good subset. Let $\vec{a}'$ be a C-basis for $\vec{a}$, and $k = |\vec{a}'| = R(\vec{a}/C)$. Then $\vec{a} \in \mathrm{acl}(C\vec{a}')$. Let $\phi(\vec{x}; \vec{y})$ be a C-formula such that $\phi(\vec{a}; \vec{a}')$ holds and $\phi(M; \vec{a}')$ is finite of size ℓ. Let $\pi : \mathbb{M}^n \to \mathbb{M}^k$ be the coordinate projection such that $\pi(\vec{a}) = \vec{a}'$. Let

$$X' = \{\vec{e} \in X : \phi(\vec{e}, \pi(\vec{e})) \text{ and } |\phi(M, \pi(\vec{e}))| \leq \ell\}$$

Let $Y' = \pi(X')$. Then $\pi : X' \to Y'$ is C-definable and has finite fibers, by definition of X'. Also, $\vec{a} \in X'$ and $\vec{a}' \in Y'$. Finally, Y' is C-definable, so $\dim(Y') \geq R(a'/C) = k$, implying that $\dim(Y') = k$. $\square$

Theorem 7.3. *Let $\{X_b\}_{b \in Y}$ be a definable family of subsets of $\mathbb{M}^n$. For each $k \in \mathbb{Z}$, the set*

$$Y_k := \{b \in Y : \dim(X_b) = k\}$$

is definable.

Proof. Let C define the family. By compactness, it suffices to show that each Y_k is $\vee$-definable over C. In other words, if $\dim(X_b) = k$, we should find a C-definable formula $\phi(y)$ such that $\phi(b)$ holds, and

$$\mathbb{M} \models \phi(b') \implies \dim(X_{b'}) = k.$$

Take a covering of X_b by "good" sets

$$X_b = X_1 \cup \cdots \cup X_m.$$

For each i let $k_i = \dim(X_i)$ and let $f_i : X_i \to Y_i$ be a definable surjection with finite fibers, where $Y_i \subseteq \mathbb{M}^k$ has $\dim(Y_i) = k$. We can find a finite tuple c from $\mathbb{M}$ and formulas $\psi_i(x, z)$, $\chi_i(x, y, z)$, $\mu_i(y, z)$ such that

- $\psi_i(\mathbb{M}, c) = X_i$.
- $\mu_i(\mathbb{M}, c) = Y_i$.
- $\chi_i(\mathbb{M}, c)$ is the graph of f_i.

Now we can write a first-order formula $\nu(b, c)$ (with hidden parameters from C) expressing that

- $b \in Y$.
- $\psi_i(\mathbb{M}, c) \subseteq X_b$ for all i

- X_b is covered by the sets $\psi_i(\mathbb{M}, c)$.
- $\mu_i(\mathbb{M}, c) \subseteq \mathbb{M}^{k_i}$ and $\dim(\mu_i(\mathbb{M}, c)) = k_i$ (possible by Lemma 7.1 and elimination of $\exists^\infty$).
- $\chi_i(\mathbb{M}, c)$ is the graph of a surjection from $\psi_i(\mathbb{M}, c)$ to $\mu_i(\mathbb{M}, c)$ with finite fibers (possible by elimination of $\exists^\infty$).

Then let $\phi(y)$ be the formula $\exists z : \nu(y, z)$. If $\phi(b')$ holds, take c' such that $\nu(b', c')$ holds. Let $X'_i = \psi_i(\mathbb{M}, c')$, let $Y'_i = \mu_i(\mathbb{M}, c')$, and let f'_i be the finite-fiber surjection from X'_i to Y'_i whose graph is $\chi_i(\mathbb{M}, c)$. Then $X'_i \rightsquigarrow Y'_i$ and $\dim(Y'_i) = k_i$ and $X_{b'} = \bigcup_{i=1}^n X'_i$. Thus

$$\dim(X_{b'}) = \max_i \dim(X'_i) = \max_i k_i = \max_i \dim(X_i) = \dim(X_b) = k.$$

So $\phi(y)$ has the desired property. □

Lemma 7.4. *If $X \rightsquigarrow Y$ and $Y \subseteq \mathbb{M}^n$ and $\dim(X) = \dim(Y) = n$, then $\deg(X) < \infty$.*

Proof. Let $f : X \to Y$ be the definable surjection with finite fibers. Let Z_i be the set of $b \in Y$ such that $|f^{-1}(b)| = i$. Then the Z_i cover Y, so almost all the Z_i are empty (compactness). Thus there is a uniform bound d on the fiber sizes $|f^{-1}(b)|$.

We claim $\deg(X) \leq d$. Otherwise, take $X_1, \ldots, X_{d+1}$ disjoint definable subsets of X with $\dim(X_i) = n$. Let $Y_i = f(X_i)$. The map $X_i \to Y_i$ has fibers of rank 0, so $\dim(Y_i) = \dim(X_i) = n$. As $\deg(\mathbb{M}^n) = 1$, we see that $\bigcap_{i=1}^{d+1} Y_i$ is non-empty. Take $b \in \bigcap_{i=1}^{d+1} Y_i$. Then $b \in f(X_i)$, so the fiber $f^{-1}(b)$ intersects each $X_1, \ldots, X_{d+1}$. Thus $|f^{-1}(b)| \geq d + 1$, a contradiction. □

Theorem 7.5. *For any definable X, $\deg(X) < \infty$.*

Proof. By Lemmas 7.2 and 7.4, X is a union of sets of finite Morley rank. By the additivity in Proposition 3.4, it follows that $\deg(X) < \infty$. □