Introduction to stability theory

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1 Where we are headed

Let M be a structure. If $\phi(\vec{x})$ is a formula,

$$\phi(M) := \{\vec{a} \in M^n : M \models \phi(\vec{a})\}$$

More generally if $\phi(\vec{x}, \vec{y})$ is a formula and $\vec{b} \in M$, then

$$\phi(M, \vec{b}) := \{\vec{a} \in M^n : M \models \phi(\vec{a}, \vec{b})\}.$$

Sets of the form $\phi(M, \vec{b})$ are called *definable sets*.

One can also talk about definable functions, definable groups, definable fields, definable group actions, etc.

A theory T is *strongly minimal* if for every model $M \models T$, every definable subset $X \subseteq M$ is finite or *cofinite* (meaning the complement $M \setminus X$ is finite).

Strong minimality is satisfied by the theory of algebraically closed fields, and appears naturally when analyzing κ-categorical theories (theories with a unique model of size κ).

Let M be a model of a strongly minimal theory T. It turns out that every definable set $X \subseteq M^n$ can be assigned a "dimension" $\dim(X) \in \{-\infty, 0, 1, 2, \ldots\}$ satisfying certain intuitive properties:

- $\dim(M^n) = n$.
- $\dim(X \times Y) = \dim(X) + \dim(Y)$
- $\dim(X \cup Y) = \max(\dim(X), \dim(Y))$.
- If $f : X \to Y$ is a definable bijection, then $\dim(X) = \dim(Y)$.
- $\dim(X) = -\infty$ iff $X = \emptyset$
- $\dim(X) \leq 0$ iff X is finite.

There is also a lot of structure on definable fields and groups. Every definable field is finite or algebraically closed. If G is a definable group, there is a sequence of definable subgroups

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

where the quotients G_{i+1}/G_i are either abelian or simple. The *Cherlin-Zilber conjecture* says that the simple groups are all algebraic groups over algebraically closed fields. The simplest known case of the conjecture is the following: if G is a simple definable group, and G acts transitively on a set of dimension 1, then $\dim(G) = 3$ and G is $PGL_2(K)$ for some $K \models ACF$.

Our main goals are

1. The dimension theory.
2. The simplest case of the Cherlin-Zilber conjecture.

2 (Strong) minimality and elementary equivalence

Definition 2.1. *An infinite structure M is* minimal *if every definable set $D \subseteq M$ is finite or "cofinite," meaning that $M \setminus D$ is finite.*

Definition 2.2. *Two structures M, M' are* elementarily equivalent*, written $M \equiv M'$, if for all sentences σ,*

$$M \models \sigma \iff M' \models \sigma.$$

Definition 2.3. *M is* strongly minimal *if every $M' \equiv M$ is minimal.*

Definition 2.4. *A theory T is* strongly minimal *if every $M \models T$ is minimal.*

Remark 2.5. *If $M \equiv M'$ and M is strongly minimal, then M' is strongly minimal.*

Remark 2.6. *If $M' \equiv M$ and $M \models T$, then $M' \models T$. Therefore, T is strongly minimal if and only if every model $M \models T$ is strongly minimal.*

Remark 2.7. *For any structure M there is a theory*

$$Th(M) = \{\sigma \in \mathcal{L} : M \models \sigma\}$$

such that

$$M' \models Th(M) \iff M' \equiv M.$$

Therefore, M is strongly minimal iff M is a model of a strongly minimal theory.

3 Bounds and $\exists^\infty$

Lemma 3.1. *Let T be strongly minimal and $\phi(x, \vec{y})$ be a formula. Then there is an integer n such that for every $M \models T$ and $\vec{b} \in M$, one of the sets $\phi(M, \vec{b})$ and $M \setminus \phi(M, \vec{b})$ has size $\leq n$.*

Proof. Otherwise, for every n there is a model M_n and tuple $\vec{b}_n \in M$ such that both the sets $\phi(M_n, \vec{b}_n)$ and $\neg\phi(M_n, \vec{b}_n)$ have size $> n$. Let $\mathcal{L}'$ be $\mathcal{L}$ plus a new constant symbol $\vec{b}$. Let $\sigma_n \in \mathcal{L}'$ be the statement asserting that there are distinct $a_1, \ldots, a_n, a'_1, \ldots, a'_n$ such that $\phi(a_i, \vec{b})$ holds and $\phi(a'_i, \vec{b})$ doesn't hold. Note that $(M_n, \vec{b}_n) \models T \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$. By compactness, there is

$$(M, \vec{b}) \models T \cup \{\sigma_1, \sigma_2, \ldots\}.$$

Then $M \models T$ and both of the sets $\phi(M, \vec{b})$ and $\neg\phi(M, \vec{b})$ are infinite, contradicting strong minimality of T. □

Proposition 3.2. *A structure M is strongly minimal if and only if M is minimal and for every formula $\phi(x; \vec{y})$ there is an n_ϕ such that for every $\vec{b} \in M$, either $\phi(M, \vec{b})$ is infinite, or $\phi(M, \vec{b})$ has size at most n_ϕ.*

Proof. The "only if" direction is Lemma 3.1.

For the "if" direction, suppose M satisfies the condition, and $M' \equiv M$. Let $m_\phi = \max(n_\phi, n_{\neg\phi})$. There is a statement τ_ϕ asserting that for every $\vec{b}$, one of the sets $\phi(M; \vec{b})$ or $\neg\phi(M; \vec{b})$ has size at most m_ϕ. Then $M \models \tau_\phi \implies M' \models \tau_\phi$. As $M' \models \tau_\phi$ for all ϕ, the structure M' is minimal. □

Theorem 3.3. *Let T be a strongly minimal theory. For every formula $\phi(\vec{x}, \vec{y})$ there is a formula $\psi(\vec{y})$ such that for any model M and tuple $\vec{b}$,*

$$M \models \exists^\infty \vec{x} : \phi(\vec{x}, \vec{b}) \iff M \models \psi(\vec{b}).$$

Thus T "eliminates $\exists^\infty$."

Proof. First suppose that $|\vec{x}| = 1$. Then there is n_ϕ such that

$$T \vdash \exists^\infty x : \phi(x, \vec{y}) \iff \exists^{\geq n_\phi} x : \phi(x, \vec{y})$$

by Lemma 3.1. (Note that $\exists^{\geq n_\phi} x : \phi(x, \vec{y})$ is essentially a first-order formula.) If $|\vec{x}| > 1$, use the fact that $X \subseteq M^n$ is infinite if and only if $\pi_i(X) \subseteq M$ is infinite for one of the coordinate projections

$$\pi_i(x_1, \ldots, x_n) = x_i.$$

Thus

$$\exists^\infty (x_1, \ldots, x_n) : P(\vec{x}) \iff \bigvee_{i=1}^{n} \exists^\infty x_i \exists (x_1, \ldots, \hat{x_i}, \ldots, x_n) : P(\vec{x}),$$

allowing us to reduce to the $|\vec{x}| = 1$ case. □

4 Elementary extensions

If M, M' are two structures, we write $M \leq M'$ if M is a substructure of M', or equivalently, M' is an extension of M.

Definition 4.1. *$M \preceq M'$ if $M \leq M'$ and for every tuple $\vec{a} \in M$ and formula $\phi(\vec{x})$,*

$$M \models \phi(\vec{a}) \iff M' \models \phi(\vec{a})$$

We say that M is an elementary substructure of M' *or M' is an* elementary extension of *M.*

If $M \preceq M'$, then $M \equiv M'$, by considering $\phi(\vec{x})$ that don't mention $\vec{x}$.

If M is a structure and $S \subseteq M$, we let $\mathcal{L}(S)$ denote the original language $\mathcal{L}$ expanded by a constant symbol for every element of S.

Fact 4.2. *Let* $\mathrm{diag}(M)$ *and* $\mathrm{el.diag}(M)$ *be the following sets of $\mathcal{L}(M)$ formulas:*

$$\mathrm{el.diag}(M) = \{\phi(\vec{a}) : \phi(\vec{x}) \in \mathcal{L},\ \vec{a} \in M,\ M \models \phi(\vec{a})\}$$
$$\mathrm{diag}(M) = \{\phi(\vec{a}) \in \mathrm{el.diag}(M) : \phi(\vec{x}) \text{ is qfree}\}$$

Then

- *models of* $\mathrm{diag}(M)$ *are equivalent to structures extending M.*
- *models of* $\mathrm{el.diag}(M)$ *are equivalent to elementary extensions of M.*

5 Types

If M is a structure and $\vec{x}$ is a tuple of variables, a *partial type* is a collection $\Sigma(\vec{x})$ of $\mathcal{L}(M)$-formulas $\phi(\vec{x})$.

An element $\vec{a} \in M$ *realizes* or *satisfies* $\Sigma(\vec{x})$, if $M \models \phi(\vec{a})$ for all $\phi(\vec{x}) \in \Sigma(\vec{x})$. We write this as $M \models \Sigma(\vec{a})$.

A partial type $\Sigma(\vec{x})$ is *consistent* or *finitely satisfiable* if every finite subtype $\Sigma_0(\vec{x}) \subseteq \Sigma(\vec{x})$ has a realization in M.

A partial type $\Sigma(\vec{x})$ is *over* a subset $S \subseteq M$ if every $\phi(\vec{x}) \in \Sigma(\vec{x})$ is an $\mathcal{L}(S)$ formula.

A *(complete) type* over $S \subseteq M$ is a maximal consistent partial type over S. By Zorn's lemma, every partial type over S extends to a (complete) type over S.

Lemma 5.1. *Let $\Sigma(\vec{x})$ be a consistent partial type over S. Then $\Sigma(\vec{x})$ is complete if and only if for every $\phi(\vec{x}) \in \mathcal{L}(S)$,*

$$\phi(\vec{x}) \in \Sigma(\vec{x}) \text{ or } \neg\phi(\vec{x}) \in \Sigma(\vec{x})$$

Proof. Suppose the criterion holds. Then $\Sigma(\vec{x})$ is maximal. Otherwise, $\Sigma'(\vec{x}) := \Sigma(\vec{x}) \cup \{\phi(\vec{x})\}$ is consistent for some $\phi(\vec{x}) \notin \Sigma(\vec{x})$. By the criterion, $\neg\phi(\vec{x}) \in \Sigma(\vec{x})$. But then

$$\{\phi(\vec{x}), \neg\phi(\vec{x})\} \subseteq \Sigma'(\vec{x}),$$

so $\Sigma'(\vec{x})$ is not consistent.

Conversely, suppose $\Sigma(\vec{x})$ is complete, but there is $\phi(\vec{x})$ over S such that neither $\phi(\vec{x})$ nor $\neg\phi(\vec{x})$ are in $\Sigma(\vec{x})$. By maximality, there is a finite subtype $\Sigma_0(\vec{x}) \subseteq \Sigma(\vec{x})$ such that neither

$$\Sigma_0(\vec{x}) \cup \{\phi(\vec{x})\} \text{ nor } \Sigma_0(\vec{x}) \cup \{\neg\phi(\vec{x})\}$$

has a realization in M. But $\Sigma_0(\vec{x})$ has a realization $\vec{a} \in M$, and then $\vec{a}$ realizes one or the other, depending on whether $M \models \phi(\vec{a})$. Contradiction. □

If $\vec{a} \in M$ and $B \subseteq M$, we let $\mathrm{tp}(\vec{a}/B)$ be the set of all $\phi(\vec{x}) \in \mathcal{L}(B)$ such that $M \models \phi(\vec{a})$. This is a consistent complete type over B.

Proposition 5.2. *Let M be a structure.*

1. *If $\Sigma(\vec{x})$ is a consistent partial type over M, then there is $M' \succeq M$ and $\vec{a} \in M'$ such that $M' \models \Sigma(\vec{a})$.*
2. *If $B \subseteq M$ and $p(\vec{x})$ is a complete type over B, there is $M' \succeq M$ and $\vec{a} \in M'$ such that $\mathrm{tp}(\vec{a}/S) = p(\vec{x})$.*

Proof. We prove the first point. Let $\mathcal{L}'$ be $\mathcal{L}(M)$ with a new constant tuple $\vec{a}$ of length $|\vec{x}|$. We claim $\Sigma(\vec{a}) \cup \mathrm{el.\,diag}(M)$ is consistent. Indeed, for any finite $\Sigma_0(\vec{x}) \subseteq \Sigma(\vec{x})$, the theory

$$\Sigma_0(\vec{a}) \cup \mathrm{el.\,diag}(M)$$

has a model, namely $(M, \vec{a})$ with $\vec{a}$ realizing Σ_0. By compactness, there is

$$(M', \vec{a}) \models \Sigma(\vec{a}) \cup \mathrm{el.\,diag}(M),$$

so $M' \succeq M$ and $M' \models \Sigma(\vec{a})$. This proves the first point, and the second point follows immediately. □

6 Quantifier elimination and the example of ACF

Definition 6.1. *A subset of M^n is qfree-definable if it is of the form $\phi(M; \vec{b})$ for some qfree formula $\phi(\vec{x}, \vec{y})$.*

Definition 6.2. *T has* quantifier elimination *(QE) if for every formula $\phi(\vec{x})$ there is a qfree formula $\phi'(\vec{x})$ such that*

$$T \vdash \forall \vec{x} : (\phi(\vec{x}) \Leftrightarrow \phi'(\vec{x}))$$

Let ACF be the theory of algebraically closed fields $(K, +, \cdot, -, 0, 1)$. We will show that ACF has quantifier-elimination and is strongly minimal.

Lemma 6.3. *Let $K \models ACF$ and F be a subfield of K, and let $D \subseteq K$ be qfree-definable over F. Then D or $K \setminus D$ is a finite subset of F^{alg}, the algebraic closure of F.*

Proof. Let $\mathcal{B}$ be the family of sets of the form S or $K \setminus S$, for S a finite subset of F^{alg}. We must show $D \in \mathcal{B}$.

We can write D as $\phi(x, \vec{b})$ for some qfree formula $\phi(x, \vec{y})$ and tuple $\vec{b} \in F$. Then $\phi(x, \vec{b})$ is a boolean combination of atomic relations of the form

$$P(x) = Q(x)$$

where P, Q are polynomials in $F[x]$. It suffices to consider atomic relations, because $\mathcal{B}$ is closed under finite boolean operations (union, intersection, complement). Moving Q to the other side, we reduce to the case

$$P(x) = 0.$$

We can factor $P(x)$ over $K[x]$, because $K \models ACF$:

$$(x - a_1)(x - a_2) \cdots (x - a_n) = 0$$

where $a_1, \ldots, a_n \in F^{alg}$. Therefore, the associated qfree-definable set is $\{a_1, \ldots, a_n\} \in \mathcal{B}$. $\square$

Therefore, ACF is strongly minimal as long as ACF has QE.

For any $K \models ACF$ and tuple $\vec{a} \in K$, let

$$\text{qftp}(\vec{a}/\emptyset) = \{\phi(\vec{x}) \in \text{tp}(\vec{a}/\emptyset) : \phi(\vec{x}) \text{ is qfree}\}.$$

Fact 6.4. *Let $K_1, K_2 \models ACF$ and $\vec{a}_i$ be a tuple in K_i for $i = 1, 2$. Let F_i be the subfield generated by $\vec{a}_i$. Then*

$$\text{qftp}(\vec{a}_1/\emptyset) = \text{qftp}(\vec{a}_2/\emptyset)$$

if and only if there is an isomorphism $F_1 \cong F_2$ carrying $\vec{a}_1$ to $\vec{a}_2$.

(This is a general model-theoretic fact that can be pedantically verified by unwinding definitions.)

Lemma 6.5. *Let $\phi(\vec{x})$ be a formula. Suppose that for any models $K_1, K_2 \models ACF$ and tuples $\vec{a}_1 \in K_1$ and $\vec{a}_2 \in K_2$,*

$$\text{qftp}(\vec{a}_1/\emptyset) = \text{qftp}(\vec{a}_2/\emptyset) \implies (K_1 \models \phi(\vec{a}_1) \iff K_2 \models \phi(\vec{a}_2))$$

Then $ACF \vdash \phi(\vec{x}) \iff \psi(\vec{x})$ for some qfree $\psi(\vec{x})$.

Proof. Suppose no such ψ exists. Let

$$\Sigma(\vec{x}) = \{\neg\psi(\vec{x}) : \psi(\vec{x}) \text{ qfree and } ACF \vdash \psi(\vec{x}) \to \phi(\vec{x})\} \cup \{\phi(\vec{x})\}$$

If $ACF \cup \Sigma(\vec{x})$ is inconsistent, there are $\psi_1, \ldots, \psi_n$ such that

$$ACF \vdash (\psi_1(\vec{x}) \vee \cdots \vee \psi_n(\vec{x})) \to \phi(\vec{x})$$
$$ACF \vdash \phi(\vec{x}) \to (\psi_1(\vec{x}) \vee \cdots \vee \psi_n(\vec{x}))$$

contradicting the assumption. So $ACF \cup \Sigma(\vec{x})$ is consistent. There is $K \models ACF$ and $\vec{a} \in K$ such that $K \models \Sigma(\vec{a})$. Now let

$$\Sigma'(\vec{x}) = \{\neg\phi(\vec{x})\} \cup \text{qftp}(\vec{a}/\emptyset)$$

We claim $\Sigma'(\vec{x}) \cup ACF$ is consistent. Otherwise, there is qfree $\psi(\vec{x}) \in \text{qftp}(\vec{a}/\emptyset)$ such that

$$ACF \vdash \psi(\vec{x}) \to \phi(\vec{x}),$$

implying $\neg\psi(\vec{x}) \in \Sigma(\vec{x})$, contradicting the fact that $K \models \Sigma(\vec{a})$. Thus $ACF \cup \Sigma'(\vec{x})$ is consistent. Take a model $K' \models ACF$ and $\vec{a}' \in K'$ with $K' \models \Sigma'(\vec{a}')$. Then

$$\text{qftp}(\vec{a}'/\emptyset) = \text{qftp}(\vec{a}/\emptyset),$$

but

$$K \models \phi(\vec{a})$$
$$K' \models \neg\phi(\vec{a}),$$

contradicting the assumption. □

Lemma 6.6. *Let $\phi(\vec{x}, y)$ be a quantifier-free formula. Then*

$$ACF \vdash (\exists y : \phi(\vec{x}, y)) \iff \psi(\vec{x})$$

for some qfree formula $\psi(\vec{x})$.

Proof. By Lemma 6.5 it suffices to show the following: if $K_1, K_2 \models ACF$, if $\vec{a}_i \in K_i$, if $\text{qftp}(\vec{a}_1/\emptyset) = \text{qftp}(\vec{a}_2/\emptyset)$, then

$$K_1 \models \exists y : \phi(\vec{a}_1, y) \iff K_2 \models \exists y : \phi(\vec{a}_2, y)$$

Let F_i be the subfield of K_i generated by $\vec{a}_i$. By Fact 6.4, there is an isomorphism $F_1 \cong F_2$ sending $\vec{a}_1$ to $\vec{a}_2$. We may assume $F_1 = F_2 = F$ and $\vec{a}_1 = \vec{a}_2 = \vec{a}$. Let R be the F-algebra $K_1 \otimes_F K_2$. It is a non-zero commutative ring, so it has at least one maximal ideal $\mathfrak{m}$, and $R/\mathfrak{m}$ is a field. Let K_3 be the algebraic closure of $R/\mathfrak{m}$. Then we have a diagram

$$\begin{array}{ccccc} F & \longrightarrow & K_1 & & \\ \downarrow & & \downarrow & & \\ K_2 & \longrightarrow & R & \longrightarrow & K_3 \end{array}$$

in the category of unital rings. The maps $K_1 \to K_3$ and $K_2 \to K_3$ are embeddings, because the kernel is a proper ideal hence 0. So we have embedded the two models $K_1, K_2 \models ACF$ over F into a third model $K_3 \models ACF$. We claim

$$K_1 \models \exists y : \phi(\vec{a}, y) \iff K_3 \models \exists y : \phi(\vec{a}, y) \tag{1}$$

As $\phi(x, y)$ is qfree, this is equivalent to

$$\exists b \in K_1 : K_3 \models \phi(\vec{a}, b) \iff \exists b \in K_3 : K_3 \models \phi(\vec{a}, b).$$

The $\Rightarrow$ direction is trivial, as $K_1 \subseteq K_3$. For the $\Leftarrow$ direction, let $D = \phi(\vec{a}, K_3)$. By Lemma 6.3, D is either S or $K_3 \setminus S$ for some finite subset $S \subseteq K_1$. Either way,

$$D \neq \emptyset \implies D \cap K_1 \neq \emptyset.$$

This proves (1). Similarly,

$$K_2 \models \exists y : \phi(\vec{a}, y) \iff K_3 \models \exists y : \phi(\vec{a}, y),$$

so we have shown

$$K_1 \models \exists y : \phi(\vec{a}, y) \iff K_2 \models \exists y : \phi(\vec{a}, y)$$

as desired. □

Theorem 6.7. *ACF has quantifier elimination and is strongly minimal.*

Proof. By Lemma 6.3, we only need to show QE. We prove by induction on $\phi(\vec{x})$ that there is qfree $\phi'(\vec{x})$ such that $ACF \vdash \phi(\vec{x}) \leftrightarrow \phi'(\vec{x})$.

If $\phi(\vec{x})$ is atomic, then $\phi(\vec{x})$ is already qfree and we can take $\phi'(\vec{x}) = \phi(\vec{x})$.

If $\phi(\vec{x}) = \psi(\vec{x}) \wedge \chi(\vec{x})$, then by induction there are qfree $\psi'(\vec{x})$ and $\chi'(\vec{x})$ such that

$$ACF \vdash \psi(\vec{x}) \leftrightarrow \psi'(\vec{x})$$
$$ACF \vdash \chi(\vec{x}) \leftrightarrow \chi'(\vec{x})$$

Then

$$ACF \vdash (\psi(\vec{x}) \wedge \chi(\vec{x})) \leftrightarrow (\psi'(\vec{x}) \wedge \chi'(\vec{x})).$$

The case of $\neg\psi(\vec{x})$ and $\psi(\vec{x}) \vee \chi(\vec{x})$ are similar. If $\phi(\vec{x}) = \exists y : \psi(\vec{x}, y)$, then by induction there is $\psi'(\vec{x}, y)$ equivalent to $\psi(\vec{x}, y)$. By Lemma 6.6,

$$\phi(\vec{x}) \approx \exists y : \psi'(\vec{x}, y) \approx \phi'(\vec{x})$$

for some qfree $\phi'(\vec{x})$. Finally, if $\phi(\vec{x})$ is $\forall y : \psi(\vec{x}, y)$, we reduce to the cases of $\neg$ and $\exists$ because

$$\phi(\vec{x}) = \forall y : \psi(\vec{x}, y) \approx \neg\exists y : \neg\psi(\vec{x}, y).$$

□

7 Exercises

Exercise 7.1. *If $M = (M, \leq, \ldots)$ is an (infinite) totally ordered structure, then M is not strongly minimal.*

Exercise 7.2. *Let $s : \mathbb{N} \to \mathbb{N}$ be $s(x) = x + 1$. Show that the structure $(\mathbb{N}, \leq, s, 0)$ has quantifier elimination and is minimal, but not strongly minimal.*

Exercise 7.3. *Find a quantifier-free formula $\psi(a, b, c)$ equivalent to*

$$\exists x \forall y : (ax^2 + bx + c)y \neq 1$$

in any algebraically closed field.