# An introduction to infinite time decidable equivalence relation theory

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# Outline

1. Equivalence Relations
2. Borel Equivalence Relation Theory
3. Infinite Time Computable Equivalence Relation Theory

# Equivalence Relation

## Definition

*A binary relation $E \subseteq X \times X$ on a set $X$ is said to be an equivalence relation if and only if for all $x, y, z \in X$:*

1. *(reflexive)* $(x, x) \in E$.
2. *(symmetric) if* $(x, y) \in E$, *then* $(y, x) \in E$.
3. *(transitive) if* $(x, y) \in E$ *and* $(y, z) \in E$, *then* $(x, z) \in E$.

We usually use $xEy$ to denote $(x, y) \in E$, for example, "Is equal to" is an equivalence relation, we use $x = y$ instead of $(x, y) \in =$.
The following are all equivalence relations:

1. "Is similar to" on the set of all triangles.
2. "Is congruent to, modulo n" on the integers.
3. "Has the same cosine" on the set of all angles.

# Examples

1. *Coset equivalence relation* Let $G$ be a group and $H \leq G$ a subgroup. Define $g_1 \sim g_2 \Leftrightarrow g_1^{-1} g_2 \in H \Leftrightarrow g_1 H = g_2 H$.
2. *Orbit equivalence relation* Let $G \curvearrowright X$ be a group action of $G$ on $X$. $x_1 \sim x_2 \Leftrightarrow \exists g \in G \; g \cdot x_1 = x_2 \Leftrightarrow \{g \cdot x_1 : g \in G\} = \{g \cdot x_2 : g \in G\}$.
3. *Vitali equivalence relation* Let $x, y \in \mathbb{R}$. Define $x E_v y \Leftrightarrow x - y \in \mathbb{Q}$. The famous applications of the Axiom of Choice is Vitali's construction of a non-Lebesgue measurable set.
4. *Measure equivalence relation* A measure $\mu$ is absolutely continuous with respect to a measure $\nu$ if $\forall A(\nu(A) = 0 \to \mu(A) = 0)$. We write $\mu \ll \nu$. Define $\mu \equiv_m \nu \Leftrightarrow \mu \ll \nu$ and $\nu \ll \mu$.

# Reduction

## Definition

*Let $E, F$ be two equivalence relations on $X, Y$ respectively, $f : X \to Y$ is a reduction from $E$ to $F$ if*

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2)$$

*for any $x_1, x_2 \in X$. We say that $E$ is reducible to $F$ and denote $E \leq F$ if there is a reduction from $E$ to $F$ .*

## Fact

*With AC, $E \leq F$ is equivalent to the cardinality of $X/E$ is less than or equal to the cardinality of $Y/F$.*

We need to impose definability requirements on the spaces and reduction functions.

# Classification Problem

1. - *Matrix similarity* Let $A, B$ be $n \times n$ matrices. Define $A \approx B \Leftrightarrow A = P^{-1}BP$, where $P$ is invertible matrice.
   - Let $J(A)$ be the Jordan normal form of $A$. Then $A \approx B \Leftrightarrow J(A) = J(B)$.
   - $\approx \leq =_{\mathbb{R}}$.
2. - *The fundamental theorem of finitely generated abelian groups* Every finitely generated abelian group $G$ is isomorphic to a direct sum
   $$\mathbb{Z}/p_1^{r_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_n^{r_n}\mathbb{Z} \oplus \mathbb{Z}^m$$
   where $n, m \leq 0$, $r_i > 0$ and $p_i$ are different primes.
   - Let $M(G) = (m, p_i, r_i)_{i=1,\ldots,n}$. Then $G \cong H \Leftrightarrow M(G) = M(H)$.
   - $\cong \leq =_{\mathbb{N}}$.

# Polish Space

## Definition

*A topological space is Polish if it is separable and completely metrizable.*

## Example

1. $\mathbb{N} = \omega$ *with the discrete topology is Polish.*
2. $\mathbb{R}$ *with the usual topology is Polish.*
3. *The Baire space* $\mathcal{N} = \omega^\omega$ *is Polish. A complete metric on* $\mathcal{N}$ *is defined by*

$$d(x, y) = \begin{cases} 0, & \textit{if } x = y \\ 2^{-n-1}, & \textit{if } n \in \omega \textit{ is the least such that } x(n) \neq y(n) \end{cases}$$

4. *The Cantor space* $2^\omega$.
5. *All compact metrizable spaces are Polish*

# Borel Reduction

**Definition**

*Let $X, Y$ be Polish spaces and $E, F$ equivalence relations on $X, Y$ respectively. $E$ is Borel reducible to $F$, denoted $E \leq_B F$, if there is a Borel reduction from $E$ to $F$.*

**Definition**

*Let $E, F$ be equivalence relations on Polish spaces $X, Y$ respectively.*

1. *$E <_B F$ if $E \leq_B F$ but not $F \leq_B E$.*
2. *$E$ and $F$ are Borel bireducible (or Borel equivalent), denoted $E \sim_B F$, if both $E \leq_B F$ and $F \leq_B E$.*
3. *$E$ and $F$ are Borel isomorphic, denoted $E \cong_B F$ , if there is a Borel bijective reduction $f : X \to Y$.*

# Examples

Review the previous examples.

**Fact**

1. *The classification problem in first example ($n \times n$ matrices) is Borel equivalent to $=_{\mathbb{R}}$.*
2. *The classification problem in second example (Finitely generated abelian groups) is Borel equivalent to $=_{\mathbb{N}}$.*

**Fact**

$=_{\mathbb{N}} <_B =_{\mathbb{R}} <_B E_v <_B \equiv_m$.

# Smooth Equivalence Relations

**Definition**

*$E$ is smooth if $E \leq_B =_{\mathbb{R}}$.*

**Theorem**

*Let $E$ be a smooth equivalence relation. Then exactly one of the following holds.*

1. *$E$ has only finitely many equivalence classes.*
2. *$E \sim_B =_{\mathbb{N}}$.*
3. *$E \sim_B =_{\mathbb{R}}$.*

$E_v$ is not smooth.

# Dichotomy Theorem

**Definition**

*The equivalence relation $E_0$ on $2^\omega$ is defined by*

$$xE_0y \Leftrightarrow \exists n \forall m \geq n \; x(m) = y(m).$$

**Theorem**

$E_0 \sim_B E_v$.

**Definition**

*Let $E$ be an equivalence relation on a Polish space $X$. If $E \subseteq X \times X$ is Borel, then we say $E$ is a Borel equivalence relation.*

**Theorem (Harrington–Kechris–Louveau)**

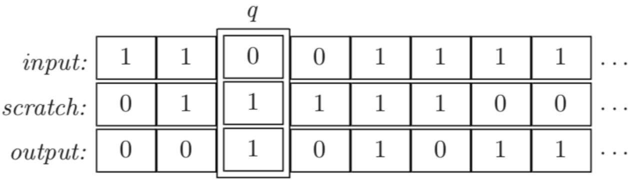
*Let $E$ be a Borel equivalence relation on a Polish space $X$. Then either $E$ is smooth or else $E_0 \leq_B E$.*

# Infinite time Turing machine

Infinite time Turing machines extend the operation of ordinary Turing machines into transfinite ordinal time. Mechanically they work just like Turing machines. What is new is the definition of the behavior of the machine at limit ordinal times.
Infinite time Turing machines were first considered by Hamkins and Kidder in 1989, and introduced by a paper of Hamkins and Lewis in 2000.

# How the machines work

An infinite time Turing machine has the same hardware as ordinary Turing machines, with a head moving on a semi-infinite paper tape, each with $\omega$ many cells exhibiting either 0 or 1, and computes according to a finite program with finitely many states.
For convenience, we have used a three tape model, with separate tapes for input, scratch work and output.



- The machine begins, like a Turing machine, with the head resting on the first cell in a special state called the *Start* state.
- At successor stages of computation, the machine operates in exactly the classical manner, according to the program instructions.
- At limit time stages, the machine is placed into the special *Limit* state, the head is reset to the leftmost cell; and the tape is updated by placing in each cell the limsup of the values previously displayed in that cell (which is the limit value, if the value had stabilized, otherwise 1).
- Computation stops only when the *Halt* state is explicitly attained, and in this case, the machines outputs the contents of the output tape.

In this way every infinite time Turing machine program $p$ determines a function. On input x, we can run the machine with program $p$, if it halts, there will be some output denote by $\varphi_p(x)$.

# Infinite time Turing machine

Since the tapes naturally accommodate infinite binary strings, the natural context for input and output to the machines is Cantor space $2^\omega$.

**Definition**

*A partial function $f : 2^\omega \to 2^\omega$ is infinite time computable (without parameters) if there is a program $p$ such that $f = \varphi_p(x)$.*

**Theorem**

*If an infinite time computation halts, then it does so in a countable ordinal number of steps.*

If a computation does not halt, then it is truly caught in an infinite loop, in the strong sense that at limits of repetitions of this loop, the computation remains inside the loop.

# The power of infinite time Turing machines

**Definition**

*A set $A$ is infinite time* decidable *if the characteristic function of $A$ is infinite time computable, and infinite time* semidecidable *if it is the domain of an infinite time computable function.*

For example, since one can simulate an ordinary Turing machine computation after $\omega$ many steps, the halting problem for ordinary Turing machines is infinite time decidable.

**Theorem**

*Every $\Pi^1_1$ set is infinite time decidable, every $\Sigma^1_1$ set is infinite time decidable.*

**Theorem**

*Every decidable set and, indeed, every semidecidable set is $\Delta^1_2$.*

# The power of infinite time Turing machines

$$\Delta^1_1 \subset \begin{array}{c}\Sigma^1_1 \\ \\ \Pi^1_1\end{array} \subset DECIDABLE \subset \begin{array}{c}SEMIDECIDABLE \\ \\ CO-SEMIDECIDABLE\end{array} \subset \Delta^1_2$$

**Theorem**

*The arithmetic sets are exactly those that are decidable in time less than $\omega^2$ and the hyperarithmetic sets are those that are decidable in time less than $\omega_1^{ck}$.*

# Compare with classical theory

By essentially the classical arguments, one can prove the infinite time analogues of the smn-theorem, the Recursion theorem and the undecidability of the infinite time halting problem (we have lightface version $h = \{p : \varphi_p(p)\downarrow\}$ and boldface version $H = \{(p, x) : \varphi_p(x)\downarrow\}$, they are not equivalent in the infinite time context).

*Warn*: a function can have a decidable graph without being a computable function.

**Theorem (Lost Melody Theorem)**

*There is a real $c$ such that the constant function $f(x) = c$ is not infinite time computable, but its graph is infinite time decidable.*

# Boldface version

The machines can be augmented with additional input tape (as an oracle in the classical manner), and doing so allows one to relativize computations to a real parameter.
We shall use from now on the following boldface analog of the infinite time computable functions. Namely,

**Definition**

*A partial function $f : 2^\omega \to 2^\omega$ is infinite time computable if there exists a $z \in 2^\omega$ such that $f$ is computed by an infinite time Turing machine with parameter $z$.*

# Boldface version

$$BOREL \subset \begin{matrix} \boldsymbol{\Sigma}^1_1 \\ \boldsymbol{\Pi}^1_1 \end{matrix} \subset C-SETS \subset DECIDABLE$$

$$\subset \begin{matrix} SEMIDECIDABLE \\ CO-SEMIDECIDABLE \end{matrix} \subset Abs\boldsymbol{\Delta}^1_2 \subset \boldsymbol{\Delta}^1_2$$

## Definition

*A set A is C-set if it belong to the smallest $\sigma$-algebra containing the Borel sets and closed under Suslin's operation $\mathcal{A}$.*
*A set A is absolutely $\boldsymbol{\Delta}^1_2$ if it is defined by a $\boldsymbol{\Pi}^1_2$ formula $\varphi$ and by a $\boldsymbol{\Sigma}^1_2$ formula $\psi$ such that the formulas $\varphi$, $\psi$ remain equivalent in any forcing extension.*

# Equivalence relation

Recall that

**Definition**

*If $E, F$ are equivalence relations on Polish spaces $X, Y$, then $E$ is Borel reducible to $F$, written $E \leq_B F$, if there is a Borel reduction from $E$ to $F$.*

Borel reducibility measures the complexity of equivalence relations as classification problems. The study of Borel reducibility is classical and highly successful.
However, there are cases of natural classifications which cannot be computed by a Borel reduction function.

# Equivalence relation

For instance, it is $\mathbf{\Delta}^1_2$ and not Borel to compute the classical Ulm invariants for a countable torsion abelian group.
One might consider $\mathbf{\Delta}^1_2$ reducibility, but the study of $\mathbf{\Delta}^1_2$ reducibility is problematic.

**Theorem**

*If $V = L$, then every infinite time decidable equivalence relation on $2^\omega$ semicomputably reduces to the equality relation.*

Under $V = L$, the infinite time semicomputable reduction theory collapses. (One should not construe that the semicomputable reduction relation is trivial, since under other hypotheses inconsistent with $\mathrm{V} = \mathrm{L}$, every semicomputable function is measurable.)

# Infinite time computable reduction

Coskey and Hamkins consider reduction functions which are computable by an infinite time Turing machine.

**Definition**

*The equivalence relation $E$ is infinite time computably reducible to $F$, written $E \leq_c F$, if there is an infinite time computable reduction from $E$ to $F$.*

Because we allow parameters, all Borel functions are infinite time computable, the infinite time computable reductions include the Borel reductions.

# Infinite time computable reduction

Classical non-reductions in the Borel theory often establish the lack of a measurable reduction.

**Theorem (Coskey, Hamkins)**

*Every infinite time computable function is a measurable function.*

Hence, many of the classical non-reductions in the Borel theory actually establish the lack of an infinite time computable reduction.
In this way, the infinite time computable reduction theory is interwoven into the classical Borel theory.

**Remark**

*The infinite time notions of reducibility are very closely related to that of absolutely* $\mathbf{\Delta}^1_2$ *reducibility, which has been treated by Hjorth and others.*

# Infinite time computable reduction

There exist natural equivalence relations which are so complex that Borel reducibility does not capture their relationship, and computable reducibility does.

**Definition**

*(1) $xE_{ck}y$ iff $x$ and $y$ can write the same ordinals in $\omega$ steps.*
*(2) $\cong_{WO}$ is the isomorphism relation restricted to the set of codes for well-orders.*

**Theorem (Coskey, Hamkins)**

*The equivalence relations $\cong_{WO}$ and $E_{CK}$ are Borel incomparable but infinite time computably bireducible.*

# Open question

However, the above example will be of high descriptive complexity.

**Question**

*Are there Borel equivalence relations $E, F$ such that $E \leq_c F$ but $E \nleq_B F$?*

Thank you!