Combinatorial implication of computability theory

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Introduction

- Many questions in computability theory, even for big question as KL-randomness vs $1-$randomness, have close connection to combinatorics.
- We present one example in this talk. We prove that the relativized version of a naturally arisen reverse math question is equivalent to a purely combinatorial question.

We thank Denis Hirschfeldt, Benoit Monin and Ludovic Patey for helpful discussion on the first example.

VWI problem

We adopt the problem-instance-solution framework to introduce the following problem. We first introduce some notations.

Definition 1 (Variable word)

An *infinite variable word* W on alphabet $\{0, \cdots, l-1\}$ is a ω-sequence of $\{0, \cdots, l-1\} \cup \{x_i : i \in \omega\}$ such that each variable x_i occurs at least once.
Given $\vec{a} = a_0 \cdots a_{k-1}$, let $W(\vec{a})$ denote the finite $\{0, \cdots, l-1\}$-string obtained by replacing x_i with a_i in W and then truncating the result just before the first occurrence of x_k.
Without loss of generality we assume that the first occurrence of x_i is smaller than that of x_{i+1} for all $i \in \omega$.

VWI problem

Example 2

Infinite variable word W on $\{0,1\}$:

$$
\begin{array}{rllllll}
 & 011 & x_0x_0\ 011 & x_1\ x_0x_0 & x_1x_1 00 & x_2x_2 \cdots & (0.1) \\
\vec{a} = 10, W(\vec{a}) = & 011 & 11\ 011 & 0\ 11 & 0000 & \cdots &
\end{array}
$$

Definition 3

- Problem: $\mathsf{VWI}(l,k)$.
- Instance: $c : l^{<\omega} \to k$.
- Solution: an infinite variable word W such that $\{W(\vec{a}) : \vec{a} \in l^{<\omega}\}$ is monochromatic.

VWI vs RCA

Joe Miller and Solomon proposed the following question in [Miller and Solomon, 2004].

Question 4

Is $\mathsf{VWI}(2, k)$ provable in RCA?

Or in terms of computability language:

Question 5

Does every computable $\mathsf{VWI}(2, k)$ instance admit computable solution?

A relativized version of the question is:

Question 6

Does every $\mathsf{VWI}(2, k)$ instance c admit c-computable solution?

Related literature

Definition 7 (VW, OVW)

If we require the occurrence of x_i being finite for all i then the problem is called VW.
If we require all the occurrence of x_i comes before any occurrence of x_{i+1} then it is called OVW (ordered variable word).

The problem is proposed by [Carlson and Simpson, 1984] and studied in [Miller and Solomon, 2004] [Liu et al., 2017]. Clearly,

Theorem 8

$\mathsf{VWI}(l,k) \le \mathsf{VW}(l,k) \le \mathsf{OVW}(l,k)$.
$\mathsf{VWI}(l,k) \Leftrightarrow \mathsf{VWI}(l,k+1), \mathsf{VW}(l,k) \Leftrightarrow \mathsf{VW}(l,k+1), \mathsf{OVW}(l,k) \Leftrightarrow \mathsf{OVW}(l,k+1)$.

Related literature

Theorem 9 ([Miller and Solomon, 2004])

There exists a computable instance of $\mathsf{OVW}(2,2)$ *that does not admit* Δ^0_2 *solution. Thus* $\mathsf{RCA}_0 + \mathsf{WKL}$ *does not prove* $\mathsf{VW}(2,2)$.

The following result answers a question of [Miller and Solomon, 2004] and [Montalbán, 2011].

Theorem 10 (Monin, Patey, L)

- *For every computable* $\mathsf{OVW}(2,k)$ *instance* c*, every* $\emptyset'$*-PA degree compute a solution to* c.
- *There exists a computable* $\mathsf{OVW}(2,2)$ *instance such that every solution is* $\emptyset'$*-DNC degree.*

Corollary 11 (Monin, Patey, L)

ACA *proves* $\mathsf{OVW}(2,k)$.

Related literature

Question 12 ([Miller and Solomon, 2004])

Does $\mathsf{OVW}(l, k)$ or $\mathsf{VW}(l, k)$ implies ACA_0 for some l?

A combinatorial equivalence of "VWI(2, 2) vs RCA"

For two sets of numbers A, B, write $A < B$ iff $\max A < \min B$.

Definition 13 ($Oppress(n_0, \cdots, n_{r-1})$)

For a sequence of integers $n_0, \cdots, n_{r-1} > 0$, let $N_0 < \cdots < N_{r-1}$ be r sets of integers with $|N_i| = n_i, i \le r-1$, let $N = \bigcup_{i \le n-1} N_i$ we say $Oppress(n_0, \cdots, n_{r-1})$ holds iff:
there exists a function $f : \mathcal{P}(N) \to \{0, 1\}$ such that for any $k \le r-1$, any $n_k + 1$ many mutually disjoint subsets $M_0, \cdots, M_{n_k}$ of N with

$$M_i \cap N_k = \{the\ i^{th}\ large\ element\ in\ N_k\} = \{\min M_i\}, 0 < i \le n_k,$$

there exists $I, J \subseteq \{1, \cdots, n_k\}$ such that:

$$f(M_0 \cup (\bigcup_{i \in I} M_i)) \neq f(M_0 \cup (\bigcup_{i \in J} M_i)).$$

A combinatorial equivalence of "VWI(2, 2) vs RCA"

Theorem 14

The following are equivalent:

- *There exists a* $\mathsf{VWI}(2,2)$ *instance* c *that does not admit* c*-computable solution.*
- *There exists an infinite sequence of positive integers* $n_0, n_1, \cdots$ *such that for all* $r \in \omega$ $Oppress(n_0, \cdots, n_r)$ *holds.*

Intuition on $Oppress(n_0, \cdots, n_{r-1})$

Suppose Φ_0^c, Φ_1^c has computed two variable word initial segment, namely W_0, W_1. For each $i \in \{0,1\}$, let $P_j^i = \{m : W_i(m) = x_j\}$, $P_0^i = \{m : W_i(m) = 1\}$. Suppose there are n_0, n_1 many variables appearing in W_0, W_1 respectively. Suppose W_1 agrees with W_0 before $|W_0|$, i.e., $|W_1| > |W_0|$, $P_0^1 \cap |W_0| = P_0^0$, $\min P_1^1 > |W_0|$.

The key note is that: if W_0 can not be extended, and for any configuration of W_0 (namely $W_0(\vec{a}), \vec{a} \in \{0,1\}^{n_0}$), $W_1/W_0(\vec{a})$ can not be extended, then $Oppress(n_0, n_1)$ holds.

We consider c as a function f : (Finite set of ω) $\times\, \omega \to \{0,1\}$ as following: $c(\sigma) = f(\sigma^{-1}(1), |\sigma|)$ and $f(B, n) = f(B \cap n, n)$ for all $B \subseteq \omega, n \in \omega$.

To see this:

To extend W_0 we need to find mutually disjoint sets $P'_i, 0 \leq i \leq n_0$ with $P'_i - P^0_i > |W_0|, i \leq n_0$ and a $p > P'_i, i \leq n_0$ such that for all $I, J \subseteq \{1, \cdots, n_0\}$: $f\left(P'_0 \cup (\bigcup_{i \in I} P'_i),\ p\right) = f\left(P'_0 \cup (\bigcup_{i \in J} P'_i),\ p\right)$.

W_0 cannot be extended implies such P'_i, p do not exist. In particular for any mutually disjoint subset $M_0, M_1, \cdots, M_{n_1}$ of n_1, let $P'_i = P^0_i \cup \big(\bigcup_{j \in M_i} P^1_j\big)$, $P'_0 = P^0_0 \cup P^1_0 \cup \big(\bigcup_{j \in M_0} P^1_j\big)$, there exists I, J with $I, J \subseteq \{1, \cdots, n_0\}$: $f\left(P'_0 \cup (\bigcup_{i \in I} P'_i),\ p\right) \neq f\left(P'_0 \cup (\bigcup_{i \in J} P'_i),\ p\right)$. Where $p = |W_1|$.

Moreover, for any configuration of W_0, $W_1/W_0(\vec{a})$ can not be extended implies for any $M_0 \subseteq \{1, \cdots, n_0\}$, let $P'_0 = P^1_0 \cup P^0_0 \cup \big(\bigcup_{j \in M_0} P^0_j\big)$, there exists $I, J \subseteq \{1, \cdots, n_1\}$ such that

$$f\left(P'_0 \cup (\bigcup_{i \in I} P^1_i),\ p\right) \neq f\left(P'_0 \cup (\bigcup_{i \in J} P^1_i),\ p\right).$$

Thus the following $\tilde{f} : \mathcal{P}(n_0 \cup n_1) \to \{0,1\}$ witness $Oppress(n_0, n_1)$:
$$\tilde{f}(M) = f\left(P_0^1 \cup P_0^0 \cup (\bigcup_{i \in M \cap n_0} P_i^0) \cup (\bigcup_{j \in M \cap n_1} P_j^1), p\right).$$

Intuition on $Oppress(n_0, \cdots, n_{r-1})$

For $\mathbf{n}, \mathbf{n}' \in \omega^{<\omega}$ we write $\mathbf{n} \leq \mathbf{n}'$ if $|\mathbf{n}| = |\mathbf{n}'|$ and $\mathbf{n}(j) \leq \mathbf{n}'(j)$ for all $j \leq |\mathbf{n}|$.
It's obvious that:

Proposition 15

For $\mathbf{n}$ *being a subsequence of* $\mathbf{n}'$, $Oppress(\mathbf{n}')$ *implies* $Oppress(\mathbf{n})$.
For $\mathbf{n} \leq \mathbf{n}'$, $Oppress(\mathbf{n})$ *implies* $Oppress(\mathbf{n}')$.

Intuition on $Oppress(n_0, \cdots, n_{r-1})$

Proposition 16

$Oppress(2,2), Oppress(2,2,2)$ holds. $Oppress(n)$ holds for all $n > 0$.

Proof.

To see $Oppress(2,2)$, consider

$$f(\rho) = \rho(0) + \rho(1) + \rho(2) \ mod \ 2.$$

To see $Oppress(2,2,2)$, consider

$$f(\rho) = I(\rho(0) + \rho(1) > 0) + \rho(2) + \rho(3) + \rho(4) \ mod \ 2.$$

Where $I()$ is the indication function.
To see $Oppress(n)$, simply consider $f(\rho) = \sum_{i<|\rho|} \rho(i) \ mod \ 2$. □

Intuition on $Oppress(n_0, \cdots, n_{r-1})$

Proposition 17

$Oppress(2,2,2,2)$ does not hold.

Proof.

We don't know the proof. Adam P. Goucher at Mathoverflow examined this using SAT solver (https://mathoverflow.net/questions/293112/ramsey-type-theorem). It's easy to check that the following functions don't work:

$$f(\rho) = I(\rho(0)+\rho(1)>0)+\rho(2)+\rho(3)+\rho(4)+\rho(6) \ mod\ 2; \quad (0.2)$$

$$f(\rho) = I(\rho(0)+\rho(1)>0)+I(\rho(2)+\rho(3)>0)+ \\ +\rho(4)+\rho(5)+\rho(6) \ mod\ 2;$$

□

Proof of theorem 14

($\Leftarrow$) Let $\mathbf{n} = n_0, n_1 \cdots$ be such an infinite sequence. Let Φ_i be all Turing functional compute a VWI solution. For simplicity reason, let's put priority aside and assume $\mathbf{n}$ is computable and all Φ_i are total. It will be clear how the proof goes without these assumptions.

Let N_0 be a set consisting n_0 many first occurrence position of variables of Φ_0;
let $N_1 > N_0$ be an arbitrary set consisting n_1 many first occurrence position of variables of Φ_1;
and let $N_2, N_3, \cdots$ be defined similarly.

For all σ with $\max N_{k+1} \geq |\sigma| > \max N_k$, define $c(\sigma)$ to be $f_k\Big(\ (N_0 \cup \cdots \cup N_k)\ \cap\ \sigma^{-1}(1)\Big)$ where f_k is the witness of $Oppress(n_0, \cdots, n_k)$.

We show that $\Phi_i = W$ is not a solution. W.l.o.g suppose N_i contains the first occurrence position of variable $x_0, \cdots, x_{n_i - 1}$,
let FO_{x_j} denote the first occurrence position of x_j in W,
let $M_0 = \{m < FO_{x_{n_i}} : W(m) = 1\} \cap (\bigcup_{l \leq i-1} N_l)$,
$M_j = \{m < FO_{x_{n_i}} : W(m) = x_j\} \cap (\bigcup_{l \geq i} N_l), j \leq n_i - 1$.
let k be such that $\max N_k < FO_{x_{n_i}} \leq \max N_{k+1}$.

Clearly $M_j \subseteq N_0 \cup \cdots \cup N_k$ are mutually disjoint with

$$M_j \cap N_i = \{\min M_j\} = \{ \text{ the } j^{th} \text{ large element of } N_i\}.$$

By definition of c and f_k, for $\vec{a} \in \{0,1\}^{n_i}$,
$c(W(\vec{a}) \upharpoonright FO_{x_{n_i - 1}}) = f_k(M_0 \cup \bigcup_{j \in \vec{a}^{-1}(1)} M_j)$. But there exists I, J with
$f_k(M_0 \cup \bigcup_{j \in I} M_j) \neq f_k(M_0 \cup \bigcup_{j \in J} M_j)$, thus there exists $\vec{a}_I, \vec{a}_J$ with
$c(W(\vec{a}_I) \upharpoonright FO_{x_{n_i - 1}}) \neq c(W(\vec{a}_J) \upharpoonright FO_{x_{n_i - 1}})$.

($\Rightarrow$) We try to construct countably many greedy solutions $\Phi_0^c, \Phi_1^c \cdots$ such that the failure of $\Phi_0^c, \Phi_1^c \cdots$ provides a sequence $\mathbf{n}$ with $Oppress(n_0, \cdots, n_r)$ holds for all r. In the following proof, we consider c as a function f : (Finite set of ω) $\times\, \omega \to \{0,1\}$ as following: $c(\sigma) = f(\sigma^{-1}(1), |\sigma|)$ and $f(B,n) = f(B \cap n, n)$ for all $B \subseteq \omega, n \in \omega$. A solution to f is a sequence of set $P_0, P_1, \cdots$ such that there exists $k \in \{0,1\}$ such that for all $I \subseteq \omega, r \in \omega$ $f(P_0 \cup (\bigcup_{j \in I} P_j), \min P_r) = k$.

Each Φ_i^c will compute a sequence of sets $P_1, P_2, \cdots$ and P_0 as the position of $x_1, x_2, \cdots$ and $\{i : W(i) = 1\}$.

Φ_0^c compute $P_1, P_2, \cdots$ as following: At the beginning, let $P_0[0] = \emptyset$ and let $P_1[0] = \{b\}$ with b arbitrary. Suppose at time t, $P_0[t], \cdots, P_n[t]$ are defined. To define P_{n+1}, try to find an integer $p_{n+1} > P_n[t]$ and mutually disjoint sets $P_j' \supseteq P_j[t], j \leq n$ with
$p_{n+1} > P_j', \quad P_j' - P_j[t] > P_n[t], j \leq n$ such that:
for all $I, J \subseteq \{1, \cdots, n\}$,

$$f\left(P_0' \cup (\bigcup_{i\in I} P_i'),\ p_{n+1}\right) = f\left(P_0' \cup (\bigcup_{i\in J} P_i'),\ p_{n+1}\right).$$

Whenever at time s such $p_{n+1}, P_j', j \leq n$ are found, update $P_j[t]$ into $P_j[s] = P_j'$ and let $P_{n+1} = \{p_{n+1}\}$.

Note that at some point t Φ_0^c can no longer find the next p_{n+1} otherwise Φ_0^c is a solution to c.

Φ_1^c will make a guess on the n that Φ_0^c can no longer find p_{n+1}. Whenever Φ_1^c find his last guess n is incorrect he destroy his current computation and do it again with a new guess $n+1$. Suppose in the end Φ_0^c output n_0 many P_j denoted as $P_j^0, j \leq n_0 - 1$. Let $m_0 = \max P_{n_0-1}^0$. Φ_1^c will act slightly different from Φ_0^c as following.

Suppose at time t, Φ_1^c has defined $P_0[t], \cdots, P_n[t] > m_0$. To define P_{n+1}, try to find an integer $p_{n+1} > P_n[t]$, a set $I \subseteq n_0$ and mutually disjoint sets $P'_j \supseteq P_j[t], j \leq n$ with $p_{n+1} > P'_j, \;\; P'_j - P_j[t] > P_n[t], j \leq n$ such that, let $\tilde{P} = \bigcup_{j \in I} P_j^0$:
for all $J, J' \subseteq \{1, \cdots, n\}$,

$$f\left(\bigcup_{i<1} P_0^i \cup P'_0 \cup \tilde{P} \cup (\bigcup_{i \in J'} P'_i), p_{n+1}\right) = f\left(\bigcup_{i<1} P_0^i \cup P'_0 \cup \tilde{P} \cup (\bigcup_{i \in J} P'_i), p_{n+1}\right).$$

Whenever at time s such $p_{n+1}, P'_j, j \leq n$ are found, update $P_j[t]$ into $P_j[s] = P'_j$ and let $P_{n+1} = \{p_{n+1}\}$.

At some point t Φ_1^c can no longer find the next p_{n+1} otherwise Φ_1^c is a solution to c. To see this, note that n_0 is finite therefore there exists $I \subseteq n_0$ such that Φ_1^c find p_n with $\tilde{P} = \bigcup_{j \in I} P_j^0$ for infinitely many n. Let $i_{-1} = 0 < i_0 < i_1 < \cdots$ and P be such that p_{i_r} is found with $\tilde{P} = P$. Let $Q_r = \bigcup_{i_{r-1} \leq j < i_r} P_j$. We have that for any $r \in \omega$, any $J', J \subseteq r$,

$$f\left((\bigcup_{i<1} P_0^i) \cup P_0 \cup P \cup (\bigcup_{j \in J'} Q_j), p_{i_r} \right) = f\left((\bigcup_{i<1} P_0^i) \cup P_0 \cup P \cup (\bigcup_{j \in J} Q_j), p_{i_r} \right),$$

and $\min Q_r = p_{i_{r-1}}$. This gives a solution to c by further thinning the sequence of sets Q_j according to the color of f.

Similarly, every Φ_i^c can only find finitely many $P_0, P_1, \cdots$. Suppose in the end Φ_i^c find $n_i > 0$ many variable sets denoted as $P_j^i, j \leq n_i - 1$. We show that $\mathbf{n} = n_0, n_1 \cdots$ is a sequence such that $Oppress(n_0, \cdots, n_r)$ holds for all r. To define f_k, the witness of $Oppress(n_0, \cdots, n_r)$, for $B \subseteq N_0 \cup \cdots \cup N_k$ let

$$f_k(B) = f\left(\bigcup_{j \leq k} P_0^j \cup (\bigcup_{r \leq k, j \in B \cap N_r} P_j^r), \max P_{n_k}^k + 1\right).$$

To see f_k witness of $Oppress(n_0, \cdots, n_r)$, let $M_0, M_1, \cdots, M_{n_i}$ be such mutually disjoint sets that

$M_j \cap N_i = \{\min M_j\} = \{$ the j^{th} large element of $N_i\}$. If for all $J, J' \subseteq n_i$, $f_k(M_0 \cup (\bigcup_{j \in J'} M_j)) = f_k(M_0 \cup (\bigcup_{j \in J} M_j))$, then it means Φ_i^c can find p_{n_i+1} with $\tilde{P} = \bigcup_{r < i, j \in M_0 \cap N_r} P_j^r, P_0' = \bigcup_{i \leq r \leq k} P_0^r$,

$P_j' = \bigcup_{r \geq i, u \in M_j \cap N_r} P_u^r$, $p_{n_i+1} = \max P_{n_k}^k + 1$.

Let $\mathcal{OPPRESS}$ denote the set of infinite sequence of integers $n_0, n_1, \cdots$ such that $Oppress(n_0, \cdots, n_r)$ holds for all r.

Theorem 18

The following two degree classes are equal:

$$\{\mathbf{c} : \mathbf{c}' \textit{ compute a member in } \mathcal{OPPRESS}.\} \tag{0.3}$$
$$\{\mathbf{c} : \mathbf{c} \textit{ compute a } \mathsf{VWI}(2,2) \textit{ instance } c \textit{ that does not admit } c\textit{-computable solution.}\}$$

On $Oppress(n_0, \cdots, n_r)$

Lemma 19

There exists a sufficiently large $R \in \omega$ such that $Oppress(\underbrace{2, \cdots, 2}_{R \text{ many}})$ does not hold.

Question 20

Does $Oppress(2, 2, 2, 3)$ holds?
Does $Oppress(2, 2, 2, R)$ holds for sufficiently large R?
Is there a sufficiently large R such that $Oppress(\underbrace{3, \cdots, 3}_{R \text{ many}})$ does not hold?

Many thanks

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