Priority Arguments

LIU Yong

Nanyang Technological University

May 10, 2019

We need a recursive list of all Turing Machines

- $\Phi^A(x)\downarrow= y$ if $\exists k, s\, \Phi_s^{A\restriction k}(x) = y$
- The least k is often denoted by $\varphi(x)$, the use function.
- $\Omega = \{\Phi \mid \Phi \text{ is a Turing Machine}\}$ is countable.
- Universal Turing Machine $\Gamma^{\square} : \omega \times \omega \to \{0, 1\}$ satisfies:

$$\forall \Phi \exists e\, \Phi^{\square}(\cdot) = \Gamma^{\square}(e, \cdot)$$

- Therefore, $\Omega = \{\Gamma^{\square}(e, \cdot) \mid e \in \omega\}$
- Now we can say the following

  Let $\{\Phi\}$ be a recursive list of all Turing Machines.

  sometimes we also use $\{\Phi_e\}_{e\in\omega}$

The degrees $\leq_T K$

- A is *recursive* if $A = \Phi$ for some Φ
- A is *recursively enumerable* if $A = \operatorname{dom}\Phi$ for some Φ
- $K = \{e \mid \Phi_e(e)\downarrow\}$

Lemma (Shoenfield Limit Lemma)

TFAE,

(i) $A \leq_T K$

(ii) There is some recursive function f such that

$$f(x, 0) = 0$$
$$A(x) = \lim_{s\to\infty} f(x, s)$$

- We often write $A_s(x) = f(x, s)$ if such f exists.
- A is n-r.e. if $|\{s \mid A_s(x) \neq A_{s+1}(x)\}| \leq n$.

Warm Up I

Given any r.e. set $W >_T \varnothing$, we build a nonrecursive r.e. set A such that A and $W \not\leq_T A$ Requirements are

$$R_\Phi : A \neq \Phi$$

$$N_\Phi : W = \Phi^A \rightarrow W = \Delta$$

Warm Up II

Given a r.e. set $B >_T \varnothing$, we build a set A such that $\varnothing <_T A <_T B$. Requirements are

$$G : A = \Gamma^B$$

$$P_\Phi : A = \Phi \to B = \Delta$$

$$N_\Phi : B = \Phi^A \to B = \Delta$$

Sacks Splitting Theorem

Given $B >_T \varnothing$, we build r.e. set A_0, A_1 such that $A_0 \oplus A_1 \equiv_T B$ and $A_i <_T B$ for $i = 0, 1$.
Requirements are the following

$$A_0 = \Gamma_0^B$$
$$A_1 = \Gamma_1^B$$
$$B = \Theta^{A_0 \oplus A_1}$$
$$N_0(\Phi) : B = \Phi^{A_0} \to B = \Delta$$
$$N_1(\Phi) : B = \Phi^{A_1} \to B = \Delta$$

prelimiaries for 2-r.e. Splitting

- Let D be a 2-r.e. set, and $\{D_s\}_{s\in\omega}$ be the approximation.
- $L(D) = \{\langle x, s\rangle \mid x \in D_s \setminus D\}$
- $L(D)$ is r.e..
- D is r.e. in $L(D)$.
- If D is r.e. in X, then $L(D) \leq_T X$
- $L(D)$ is not well-defined on degrees.

2-r.e. splitting theorems

Let $B >_T \varnothing$ be a 2-r.e. set and C be a Δ_2 set. Then there are 2-r.e. set A_0, A_1 such that $D \equiv_T A_0 \oplus A_1$ and $C \not\leq_T A_i$ for $i = 0, 1$. Requirements are the following

$$
\begin{aligned}
A_0 &= \Gamma_0^B \\
A_1 &= \Gamma_1^B \\
B &= \Theta^{A_0 \oplus A_1} \\
N_0(\Phi) : C &= \Phi^{A_0} \to C = \Delta^{L(B)} \\
N_1(\Phi) : C &= \Phi^{A_1} \to C = \Delta^{L(B)}
\end{aligned}
$$

Research Topics

We have many questions for this research line.

Research Topics

We have many questions for this research line.

- Can we split any 3-r.e. set?
- Can we split any 4-r.e. set?
- Can we split any 5-r.e. set?
- $\cdots$