Projective Singletons

Yizheng Zhu

University of Chinese Academy of Sciences

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Ongoing joint work with Sy D. Friedman and Sandra Müller.

Theorem (Solovay)

Assume there is a measurable cardinal, then there is a canonical Π^1_2 singleton, called $0^\#$.

Definition

x is a Π^1_2 singleton iff $\{x\}$ is a Π^1_2 set. That is, iff there is a first formula $\varphi(v, y, z)$ such that

$$v = x \text{ iff } \forall y\ \exists z\ (\omega; 0, 1, +, \cdot, v, y, z) \models \varphi(v, y, z).$$

$0^{\#}$

From a measurable cardinal, we get a nontrivial elementary embedding $j : V \to M$. Restrict j to L.

$$j \restriction L : L \to L$$

is a nontrivial elementary embedding of L into itself. Its existence is a large cardinal notion.
If there is a measurable cardinal, then

$$0^{\#} = \{\ulcorner \varphi(v_1, \ldots, v_n) \urcorner : n \in \omega \wedge L \models \varphi(\aleph_1, \ldots, \aleph_n)\}.$$

$0^\#$

Definition

$0^\#$ exists iff there is a mouse $(L_\alpha; \in, U)$.

L_α has a largest cardinal κ. U is a partial normal measure on κ. U measures all the subsets of κ in L_α and is amenable to L_α.
$(L_\alpha; \in, U)$ can be iterated.

$$j : (L_\alpha; \in, U) \to (L_\beta; \in, U')$$

The Π^1_2 definition of $0^\#$ is:

$0^\#$ codes a mouse N, N is the Σ_1 Skolem hull of $\emptyset$, and for any countable ordinal γ, the γ-th iterate of the mouse is wellfounded.

$0^\#$ and L

$0^\#$ looks like the least member that transcends L.
$0^\# \notin L$.
An inner model without $0^\#$ is close to L.

Theorem (Jensen)

If N is an inner model and $0^\# \notin N$, then for any N-cardinal $\kappa \geq \aleph_1^N$, $(\kappa^+)^N = (\kappa^+)^L$.

Question

Is there a Π^1_2 singleton x such that $0 <_L x <_L 0^\#$?

Definition

$x \leq_L y$ iff $x \in L[y]$. $x <_L y$ iff $x \leq_L y \wedge y \not\leq_L x$.

A Π^1_2 singleton between 0 and $0^\#$

Theorem (Friedman)

There is a Π^1_2 singleton x such that $0 <_L x <_L 0^\#$.

Note: This is different from the Π^1_1 case. Kleene's $\mathcal{O}$ is a Π^1_1 singleton, but there is no Π^1_1 singleton x such that $0 <_{HYP} x <_{HYP} \mathcal{O}$.

We try to generalize to the projective levels. Assume PD. The odd levels look alike.

Theorem (Kechris-Martin-Solovay)

There is a Π^1_{2n+1} singleton $M^\#_{2n-1}$ such that no Π^1_{2n+1} singleton x satisfies $0 <_{M_{2n-1}} x <_{M_{2n-1}} M^\#_{2n-1}$.

The higher degree notion

$M_n(x)$ is the least canonical inner modeel containing x and has n Woodins cardinals.

Definition

$x \leq_{M_n} y$ iff $x \in M_n(y)$. $x <_{M_n} y$ iff $x \leq_{M_n} y \wedge y \nleq_{M_n} x$.

Question

Does Friedman's result on Π^1_2 singletions generalize to Π^1_{2n}? Is there a Π^1_4 singleton x such that $0 <_{M_2} x <_{M_2} M_2^{\#}$?

We have a pseudo result and an approach towards a true result.

The Π^1_2 singleton argument

Use Jensen coding over L. Define over L a class forcing $\mathbb{P}$ with a unique $\mathbb{P}$-generic, coded by a real $R <_L 0^\#$. The $\mathbb{P}$-generic is determined by R in the following way:

- There is a Σ_1 over L class function $\langle \alpha_1, \ldots, \alpha_n \rangle \mapsto \tau(\alpha_1, \ldots, \alpha_n)$ such that the $\mathbb{P}$-generic determined by R is

$$G_R = \{p \in \mathbb{P} : p \text{ is compatible with } \tau(i_1, \ldots, i_n) \\ \text{for all indiscernibles } i_1 < \cdots < i_n\}.$$

- R codes $A \subseteq L$, Jensen coding.
- (The Π^1_2 definition of R) Whenever $\tau(\alpha_1, \ldots, \alpha_n)$ "contradicts" R (think of $(\alpha_1, \ldots, \alpha_n)$ as a guess of indescernibles), A "kills" $(\alpha_1, \ldots, \alpha_n)$ by adding a club in α_1.

The Π^1_2 singleton argument

How to construct such a $\mathbb{P}$-generic over L?
We would like $G \subseteq \mathbb{P}$ to be preserved under all embeddings of L arised from shifting indiscernibles.
If $j : L \to L$ and $j(c_i) = c_{f(i)}$, where f is an order preserving map, $(c_i : i \in \mathrm{Ord})$ is the class of indiscernibles, then $j''G \subseteq G$.

The main difficulties

We would like to force over M_2 so that $M_2[R]$ looks like $M_2(R)$. This is not a problem with the L case.
There is a technique developed by Friedman that preserves the Woodin cardinals by Jensen coding. Many extenders in M_2 prolongs to $M_2[R]$. In $M_2[R]$, there are two Woodin cardinals, and $M_2[R]$ can construct a version of $M_2(R)$.
This approach ends up with a pseudo-Π^1_4 singleton. We get a real R such that $0 <_{M_2} R <_{M_2} M_2^{\#}$ and for a Π_1-formula φ, R is the unique solution to

$$M_2[x] \models \varphi(x).$$

There is a distinction between $M_2[x]$ and $M_2(x)$.

Another approach

Instead of working with M_2, we work with the direct limit of all countable iterates of M_2, called $M_{2,\infty}$.

Theorem (Steel)

$L_{\delta^1_3}[T_3, x]$ is an initial segment of $M_{2,\infty}(x)$.

$L[T_3]$ has a higher level analog of the L-indiscernibles. There is a higher level analog of the elementary embeddings of L arised from shifting indiscernibles.
If $j : L[T_3] \to L[T_3]$ is arised from shifting "level-3 indiscernibles", then $j''G \subseteq G$.
This is possible. There is a method of characterizing $L_{\delta^1_3}[T_3]$ as a direct limit indexed by ordinals in δ^1_3 (instead of indexing by arbitrary iterates of M_2).

Thank you for your attention!