Lifting argument for Neeman's forcing with side condition

Liuzhen Wu

Institute of Mathematics
Chinese Academy of Sciences

October 19, 2018
Fudan Logic Workshop 2018

Forcing method

In 1960's, Paul Cohen introduces the forcing method into set theory. After 50 year's development, the forcing method has become one of the fundamental tools in modern set theory.

In modern treatment, the method starts from a ZFC model V named as ground model and a predesigned partial order $(P, <)$ in V. One subsequently adds a metamathematical object $G \subseteq P$ named as generic filter and defines a structure $V[G]$ named as generic extension using V and G. It could be verified that $V[G]$ is a ZFC models and satisfies some *ad-hoc* property imposed by the order P.

Proper forcing

Inspired by Laver's proof of the consistency of Borel Conjecture, Shelah isolates the following important subclass of forcing poset.

Definition (Proper forcing)

- A condition $p \in P$ is a (M, P)- generic condition if it forces that $G \cap M$ meets all the dense subset of P in M.
- The poset P is proper for $M \ni P$, if every condition in $M \cap P$ can be extended to a (M, P) generic condition.
- The poset P is proper if for club many $M \prec H(\theta)$ for some sufficiently large θ, P is proper for M.

Proper forcing and Proper forcing axiom

Typical examples of Proper forcing are ccc forcing and countable closed forcing. Like the Continuum Hypothesis, the proper forcing and its forcing axiom PFA mostly serve as proxies in the application of set theory. This phenomenon is best explained by the following results.

Theorem (Moore)

PFA implies that there is a 5-element base for uncountable linear ordering.

Theorem (Farah)

PFA implies that all automorphisms of Calkin algebra are inner.

Forcing using models as side condition

The proof of both Moore and Farah's result involves a special type of proper forcing poset introduced by Todorcevic, the forcing using models as side condition.

The typical configuration of forcing using models as side condition is as following:

- The condition p is a pair $\langle N_p, f_p \rangle$.
- N_p is a finite continuous sequence $\langle N_i \mid i \in length(N_p) \rangle$ of elementary submodels of some prefixed structure $\langle H(\theta), \in, \lhd \rangle$. N_p is usually refer to side condition.
- f_p is a partial function with domain N_p such that $f_p(N_i) \in N_{i+1}$ for all $i \in length(N_p)$. The f_p is usually called as working part.
- Ordered by reverse inclusion.

The prototype of side condition forcing is Baumgertner's forcing poset which adds a club of ω_1 using finite condition.

Side condition forcing at ω_2

A central problem regarding the forcing method is to develop the forcing theory at level at ω_2. As a test question, one of the major open problem is the following:

Question

It is consistent that all ω_2-dense suborders of reals are order isomorphic?

Baumgarterner proves that it is consistent that all ω_1-dense suborders of reals are order isomorphic. This subsequently serves as one of the 5 base element for uncountable linear ordering under PFA. It is extreme interesting and challenging to generalize this result to the ω_2 level. Following work of Neeman, it turns out that the side forcing technique is the key for the generalization.

Side condition forcing at ω_2, the development

The prototype of side condition forcing at ω_2 is the following forcing construction obtained independently by Friedman and Mitchell.

Theorem (Friedman, Mitchell)

There is a finite condition forcing adding a club at ω_2.

Using this poset, Mitchell also proves that

Theorem (Mitchell)

Con(ZFC+Mahlo cardinal exists) implies $Con(ZFC + I(\omega_2) = NS_{\omega_2} \restriction Cof(\omega_1))$.

Using the language of models as side condition, Krueger and Neeman independently develop two frameworks for side condition forcing at ω_2. The Friedman-Mitchell forcing can be viewed as special cases in their framework. In our talk, we will focus on Neeman's framework, namely models as side condition using models of two type.

Neeman's side condition forcing using models of two type

The forcing P_{Nee} involves three regular cardinals $\kappa \leq \lambda < \eta$. For simplicity, we will mainly deal with the case that $\kappa = \omega$. Let K be the structure $\langle H(\eta), \in \lhd \rangle$.

(Transitive type) Let T be the collection of all transitive structure $M \prec K$.
(Small type) Let S be the collection of $M \prec K$ such that $|M| < \lambda$ and $\kappa \subseteq M$.

Definition

The forcing P_{Nee} is defined as follows:

- A condition is a sequence $\langle M_\xi \mid \xi < \gamma \rangle$ with $\gamma < \kappa$ such that
  1. For each ξ, $M_\xi \in T \cup S$.
  2. $M_\xi \in M_{\xi+1}$ when $M_{\xi+1}$ for $\xi + 1 < \gamma$.
  3. The sequence is closed under intersections.
- Ordered by reverse inclusion.

P_{Nee} satisfies the following generalization of properness.

Lemma

Suppose $M = N \cap K$ with $N \prec (H(\theta), P)$ for sufficiently large θ. Let p be a condition such that $M = M^p_\xi$ for some ξ. Then p is (N, P_{Nee})-strongly generic.

Definition

- A condition $p \in P$ is a (M, P)-strongly generic condition if it forces that $G \cap M$ is generic for $P \cap M$. In other words it forces that $P \cap M$ is a complete subforcing of P.
- The poset P is strongly proper for $M \ni P$, if every condition in $M \cap P$ can be extended to a (M, P)-strongly generic condition.

Corollary

P_{Nee} preserves the cardinality of λ and η. In fact, $\lambda = \omega_1^{V[G]}$ and $\eta = \omega_2^{V[G]}$.

Forcing and Large cardinal

It is natural to consider forcing in presence of large cardinals. For instance, the proof of the consistency of PFA requires a large cardinal called supercompact cardinal. One of the fundamental problem in this area is to study the preservation of large cardinal properties under forcing. A typical configuration of the problem is as following:

- κ is a large cardinal, witnessing by an elementary embedding $j : V \to M$ with κ being the critical point.
- $V[G]$ is a generic extension of V, where $G \subseteq P$ is the generical filter.
- Does κ remain a large cardinal with the same strength? In particular, is there a embedding $j^+ : V[G] \to M^+$ such that $j^+ \upharpoonright V = j$. The embedding j^+ is usually refer to the lifting of j.

Equivalently, we can use the following criterion introduced by Silver.

Fact

The following are equivalent:

1. *j has a lifting j^+.*
2. *There is a $j(P)$ generic filter H over M such that $j''G \subset H$.*

Proper forcing and lifting argument

An easy observation reveals that verification of properness and the existence of lifting are very similar in many aspect. In fact, the arguments are identical for ccc and countable closed forcing.

Proposition

If $j(P)$ is strongly proper for $j''H(\theta)$ for some sufficiently large θ, then j has a lifting j^+. Here notice that $j''H(\theta) \prec j(H(\theta)) = (H(j(\theta)))^M$.

Neeman's forcing and lifting argument

Using the above fact, one can verify that P_{Nee} preserves the large cardinal property of η. In this situation, η is the critical of j. In $j(P_{Nee})$, the condition $\langle H(\theta)\rangle$ is a $(j''(H(\theta), j(P_{Nee}))$-strongly condition. Moreover, we can check that $j(P_{Nee})$ is strongly proper for $j''(H(\theta)$. Thus the lifting j^+ can always be defined. As an application, Neeman shows the following:

Corollary

Assume $\kappa = \omega$, $\lambda = \omega_1$, η is weakly compact. Then P_{Nee} forces tree property at $\omega_2^{V[G]} = \eta$.

Generalizing this proof, Holy-Lucke-Njegomir obtain new characterizations for various large cardinal via their combinatorial properties after forcing with P_{Nee}. For instance, η is supercompact if and only if P_{Nee} forces the super tree property holds at $\omega_2^{V[G]} = \eta$.

When λ is a large cardinal

The preservation of large cardinal property of λ is more subtle. For η, we use the fact that $j''H(\eta) = H(\eta)$, as $\eta = crit(j)$. Hence we can treat $j''H(\eta)$ as a T-type model. If $\lambda = crit(j)$, then $j''H(\eta)$ is no longer a T-type model in M. Nevertheless, if $j(\lambda) > \eta$ and $M^\eta \subseteq M$, then we can still view $j''H(\eta)$ as a S-type model. This gives the following criterion for the existence of lifting.

Proposition

Suppose $j(\lambda) > \eta$ and $M^\eta \subseteq M$. Then j has a lifting j^+.

When λ is huge

Due to the nature of P_{Nee}. We are mainly interested in the case when λ is a huge cardinal witnessing by j such that $j(\lambda) = \eta$. This particular case is not covered by the previous criterion. The key issue here is that $j''H(\eta)$ is neither T-type nor S-type. This inspires us to include a new type of models witnessing the property of $j''H(\eta)$.

(Intermediate type) Let I be the collection of $M \prec K$ such that $ot(M \cap \eta) = \lambda$ and for unbounded many $\delta \in M$, $M \cap H(\delta) \in S$.

Definition

The forcing $P_{NeeHuge}$ is defined as follows:

- A condition is a pair $\langle M_p, f_p \rangle$ such that
  1. $\vec{M}_p$ is a $\in$-chain of models of t-type or s-type. For any $X \in M_p$, we denote X^+ to be the least model in $\vec{M}_p$ above X and X^- to be the large model in $\vec{M}_p$ below X.
  2. f_p is a partial function with domain being the t-type models in $\vec{M}_p$. The value of $f_p(X)$ is either $\emptyset$ or a model N of i-type with $\sup(N \cap \lambda) = \sup(M \cap \lambda)$. Let $\vec{N}_p$ be the range of f_p excluding $\{\emptyset\}$.
  3. For M such that $f_p(M)$ is i-type, $M^- \in f_p(M)$ exists and is of t-type with $f_p(M^-) = \emptyset$, also if M^+ exists then $f_p(M) \in M^+$.
  4. If X_0, X_1 appears in M_p or $\lambda_{X_0} > \lambda_{X_1}$ with X_0 i-type, X_1 t-type, then $X_0 \cap X_1 \in M_p$.
- Ordered by reverse inclusion for M_p and f_p, respectively.

Similar to P_{Nee}, we are able to prove the strongly genericity for I-type model.

Lemma

Suppose $M = N \cap K$ with $N \prec (H(\theta), P)$ for sufficiently large θ. Suppose M is I-type. Let p be a condition such that $M = f_p(M')$ for some M'. Then p is (N, P_{Nee})-strongly generic.

Lemma

$j(P_{NeeHuge})$ is strongly proper for $j''H(\theta)$ for sufficiently large θ.

Corollary

$P_{NeeHuge}$ preserves the cardinality of λ and η. In fact, $\lambda = \omega_1^{V[G]}$ and $\eta = \omega_2^{V[G]}$.

Comparison between Kunen's forcing and $P_{NeeHuge}$

The hugeness is first used by Kunen's pioneering work regarding generic large cardinals. Using sophisticated forcing techniques, Knuen first prove the consistency of the existence of saturated ideal on ω_1.

Theorem

Assume the existence of a huge cardinal, there is a forcing poset forces that GCH, there is a saturated ideal on ω_1 and Chang's conjecture holds.

Lemma

Assume the existence of a huge cardinal, $P_{NeeHuge}$ forces $2^\omega = \omega_2$ and there is a presaturated ideal on ω_1 and Chang's conjecture holds.

One level up

It is natural to consider the iteration of P_{Nee}. We indicate κ, λ, η in the subscript.

Theorem (Neeman)

If θ_0 is supercompact, $\theta_1 > \theta_0$ is weakly compact. Then $P_{Nee,\omega,\omega_1,\theta_0} * P'_{Nee,\omega_1,\theta_0,\theta_1}$ forces tree properties for both ω_2 and ω_3. Here $P'_{Nee,\omega_1,\theta_0,\theta_1}$ is a variation of $P_{Nee,\omega_1,\theta_0,\theta_1}$.

If θ_0 is 2-huge with $j(\theta_0) = \theta_1$ and $j(\theta_1) = \theta_2$. It is unclear now whether $P_{NeeHuge,\omega,\theta_0,\theta_1} * P'_{NeeHuge,\theta_0,\theta_1,\theta_2}$ forces $(\omega_3, \omega_2, \omega_1) \to (\omega_2, \omega_1, \omega_0)$.