The Ultimate-L Conjecture

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Definition

Suppose λ is an uncountable cardinal.

- λ is a **singular cardinal** if there exists a cofinal set $X \subset \lambda$ such that $|X| < \lambda$.
- λ is a **regular cardinal** if there does not exist a cofinal set $X \subset \lambda$ such that $|X| < \lambda$.

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Lemma (Axiom of Choice)

Every (infinite) successor cardinal is a regular cardinal.

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Lemma (Axiom of Choice)

Every $($infinite$)$ successor cardinal is a regular cardinal.

Definition

Suppose λ is an uncountable cardinal. Then $\mathrm{cof}(\lambda)$ is the minimum possible $|X|$ where $X \subset \lambda$ is cofinal in λ.

- $\mathrm{cof}(\lambda)$ is always a regular cardinal.
- If λ is regular then $\mathrm{cof}(\lambda) = \lambda$.
- If λ is singular then $\mathrm{cof}(\lambda) < \lambda$.

The Jensen Dichotomy Theorem

Theorem (Jensen)

Exactly one of the following holds.

(1) *For all singular cardinals γ, γ is a singular cardinal in L and*
$$\gamma^+ = (\gamma^+)^L.$$
- *L is* **close** *to V.*

(2) *Every uncountable cardinal is a regular limit cardinal in L.*
- *L is* **far** *from V.*

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A strong version of Scott's Theorem:

Theorem (Silver)

Assume that there is a measurable cardinal.

- *Then L is far from V.*

Tarski's Theorem and Gödel's Response

Theorem (Tarski)

Suppose $M \models \mathrm{ZF}$ and let X be the set of all $a \in M$ such that a is definable in M without parameters.

- *Then X is not definable in M without parameters.*

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Theorem (Gödel)

Suppose that $M \models \mathrm{ZF}$ and let X be the set of all $a \in M$ such that a is definable in M from b for some ordinal b of M.

- *Then X is Σ_2-definable in M without parameters.*

Gödel's transitive class HOD

- Recall that a set M is transitive if every element of M is a subset of M.

Definition

HOD is the class of all sets X such that there exist $\alpha \in \mathrm{Ord}$ and $M \subset V_\alpha$ such that

1. $X \in M$ and M is transitive.
2. Every element of M is definable in V_α from ordinal parameters.

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2. Every element of M is definable in V_α from ordinal parameters.

- (ZF) The Axiom of Choice holds in HOD.
- $L \subseteq \mathrm{HOD}$.
- HOD is the union of all transitive sets M such that every element of M is definable in V from ordinal parameters.
  - By Gödel's Response.

Stationary sets

Definition

Suppose λ is an uncountable regular cardinal.

1. A set $C \subset \lambda$ is **closed and unbounded** if C is cofinal in λ and C contains all of its limit points below λ:
   - For all limit ordinals $\eta < \lambda$, if $C \cap \eta$ is cofinal in η then $\eta \in C$.

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2. A set $S \subset \lambda$ is **stationary** if $S \cap C \neq \emptyset$ for all closed unbounded sets $C \subset \lambda$.

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2. A set $S \subset \lambda$ is **stationary** if $S \cap C \neq \emptyset$ for all closed unbounded sets $C \subset \lambda$.

Example:

- Let $S \subset \omega_2$ be the set all ordinals α such that $\mathrm{cof}(\alpha) = \omega$.
  - S is a stationary subset of ω_2,
  - $\omega_2 \backslash S$ is a stationary subset of ω_2.

The Solovay Splitting Theorem

Theorem (Solovay)

Suppose that λ is an uncountable regular cardinal and that $S \subset \lambda$ is stationary.

- *Then there is a partition*

$$\langle S_\alpha : \alpha < \lambda \rangle$$

of S into λ-many pairwise disjoint stationary subsets of λ.

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But suppose $S \in \mathrm{HOD}$.

- Can one require

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- Can one require

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  for all $\alpha < \lambda$?

- Or just find a partition of S into 2 stationary sets, each in HOD?

Lemma

Suppose that λ is an uncountable regular cardinal and that:

- $S \subset \lambda$ *is stationary.*
- $S \in \mathrm{HOD}$.
- $\kappa < \lambda$ *and* $(2^\kappa)^{\mathrm{HOD}} \geq \lambda$.

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of S into κ-many pairwise disjoint stationary subsets of λ such that

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Then there is a partition

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of S into κ-many pairwise disjoint stationary subsets of λ such that

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But what if:

- $S = \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \omega\}$ and $(2^\kappa)^{\mathrm{HOD}} < \lambda$?

Definition

Let λ be an uncountable regular cardinal and let

$$S = \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \omega\}.$$

Then λ is ω-**strongly measurable in** HOD if there exists $\kappa < \lambda$ such that:

1. $(2^{\kappa})^{\mathrm{HOD}} < \lambda$,
2. there is no partition $\langle S_\alpha \mid \alpha < \kappa \rangle$ of S into stationary sets such that

   $$S_\alpha \in \mathrm{HOD}$$

   for all $\alpha < \lambda$.

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   for all $\alpha < \lambda$.

Lemma

Assume λ is ω-strongly measurable in HOD. *Then*

$$\mathrm{HOD} \models \lambda \textit{ is a measurable cardinal.}$$

Extendible cardinals

Lemma

Suppose that

$$\pi : V_{\alpha+1} \to V_{\pi(\alpha)+1}$$

is an elementary embedding and π is not the identity.

- *Then there exists an ordinal η that $\pi(\eta) \neq \eta$.*

- $\mathrm{CRT}(\pi)$ denotes the least η such that $\pi(\eta) \neq \eta$.

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Definition (Reinhardt)

Suppose that δ is a cardinal.

- Then δ is an **extendible cardinal** if for each $\lambda > \delta$ there exists an elementary embedding

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

  such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$.

Extendible cardinals and a dichotomy theorem

Theorem (HOD Dichotomy Theorem (weak version))

Suppose that δ is an extendible cardinal. Then one of the following holds.

(1) **No** *regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.*

Further, suppose γ is a singular cardinal and $\gamma > \delta$.

- *Then γ is singular cardinal in HOD and $\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$.*

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(2) **Every** *regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.*

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Further, suppose γ is a singular cardinal and $\gamma > \delta$.

- *Then γ is singular cardinal in* HOD *and $\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$.*

(2) **Every** *regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD*.*

- If there is an extendible cardinal then HOD must be either **close** to V or HOD must be **far** from V.
- This is just like the Jensen Dichotomy Theorem but with HOD in place of L.

Supercompactness

Definition

Suppose that κ is an uncountable regular cardinal and that $\kappa < \lambda$.

1. $\mathcal{P}_\kappa(\lambda) = \{\sigma \subset \lambda \,\big|\, |\sigma| < \kappa\}$.
2. Suppose that $U \subseteq \mathcal{P}\left(\mathcal{P}_\kappa(\lambda)\right)$ is an ultrafilter.
   - U is **fine** if for each $\alpha < \lambda$,
   $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \,\big|\, \alpha \in \sigma\} \in U.$$

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   - U is **fine** if for each $\alpha < \lambda$,
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in \sigma\} \in U.$$
   - U is **normal** if for each function
     $$f : \mathcal{P}_\kappa(\lambda) \to \lambda$$
     such that
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid f(\sigma) \in \sigma\} \in U,$$
     there exists $\alpha < \lambda$ such that
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid f(\sigma) = \alpha\} \in U.$$

Supercompact cardinals

Definition (Solovay, Reinhardt)

Suppose that κ is an uncountable regular cardinal.

- Then κ is a **supercompact cardinal** if for each $\lambda > \kappa$ there exists an ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that:
  - U is κ-complete, normal, fine ultrafilter.

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Lemma (Magidor)

Suppose that δ is strongly inaccessible. Then the following are equivalent.

(1) *δ is supercompact.*

(2) *For all $\lambda > \delta$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and an elementary embedding*

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that $\mathrm{CRT}(\pi) = \bar{\delta}$ and such that $\pi(\bar{\delta}) = \delta$.

Supercompact cardinals and a dichotomy theorem

Theorem

Suppose that δ is an supercompact cardinal, $\kappa > \delta$ is a regular cardinal, and that κ is ω-strongly measurable in HOD.

- *Then every regular cardinal $\lambda > 2^{\kappa}$ is ω-strongly measurable in HOD.*

Supercompact cardinals and a dichotomy theorem

Theorem

Suppose that δ is an supercompact cardinal, $\kappa > \delta$ is a regular cardinal, and that κ is ω-strongly measurable in HOD.

- *Then every regular cardinal $\lambda > 2^\kappa$ is ω-strongly measurable in HOD.*

- Assuming δ is an extendible cardinal then one obtains a much stronger conclusion.

Supercompact cardinals and the Singular Cardinals Hypothesis

Theorem (Solovay)

Suppose that δ is a supercompact cardinal and that $\gamma > \delta$ is a singular strong limit cardinal.

- *Then $2^\gamma = \gamma^+$.*

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Theorem (Silver)

Suppose that δ is a supercompact cardinal. Then there is a generic extension $V[G]$ of V such that in $V[G]$:

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- Solovay's Theorem is the strongest possible theorem on supercompact cardinals and the Generalized Continuum Hypothesis.

The δ-cover and δ-approximation properties

Definition (Hamkins)

Suppose N is an inner model and that δ is an uncountable regular cardinal of V.

1. N has the δ-**cover property** if for all $\sigma \subset N$, if $|\sigma| < \delta$ then there exists $\tau \subset N$ such that:
   - $\sigma \subset \tau$,
   - $\tau \in N$,
   - $|\tau| < \delta$.

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2. N has the δ-**approximation property** if for all sets $X \subset N$, the following are equivalent.
   - $X \in N$.
   - For all $\sigma \in N$ if $|\sigma| < \delta$ then $\sigma \cap X \in N$.

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For each (infinite) cardinal γ:

- $H(\gamma)$ denotes the union of all transitive sets M such that $|M| < \gamma$.

The Hamkins Uniqueness Theorem

Theorem (Hamkins)

Suppose N_0 and N_1 both have the δ-approximation property and the δ-cover property. Suppose

- $N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+)$.

Then:

- $N_0 = N_1$.

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Then:

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Corollary

Suppose N has the δ-approximation property and the δ-cover property. Let $A = N \cap H(\delta^+)$.

- *Then $N \cap H(\gamma)$ is (uniformly) definable in $H(\gamma)$ from A,*
  - *for* **all** *strong limit cardinals $\gamma > \delta^+$.*

- N is a Σ_2-definable class from parameters.

Inner models with the δ-approximation property and the δ-cover property are close to V

Theorem

Suppose N is an inner model with the δ-approximation property and the δ-cover property.

- *Suppose $\gamma > \delta$ and γ is a singular cardinal.*

Then:

- *γ is a singular cardinal in N.*
- *$\gamma^+ = (\gamma^+)^N$.*

Set Theoretic Geology

Definition (Hamkins)

An inner model N is a **ground** of V if

- $N \models \mathrm{ZFC}$.
- There is a partial order $\mathbb{P} \in N$ and an N-generic filter $G \subseteq \mathbb{P}$ such that $V = N[G]$.
  - G is allowed to be trivial in which case $N = V$.

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Lemma (Hamkins)

Suppose N is a ground of V. Then for all sufficiently large regular cardinals δ:

- *N has the δ-approximation property.*
- *N has the δ-cover property.*

- Simply take δ be any regular cardinal of N such that $|\mathbb{P}|^N < \delta$.

Corollary

The grounds of V are Σ_2-definable classes from parameters.

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What is the possible structure of the grounds of V?

- This is part of the first order theory of V.

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What is the possible structure of the grounds of V?

- This is part of the first order theory of V.
- Suppose $N \subseteq M \subseteq V$, N is a ground of V, and $M \models \mathrm{ZFC}$.
  - Then M is a ground of V and N is a ground of M.

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The **mantle** of V is the intersection of all the grounds of V.

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Let $\mathbb{M}$ be the mantle of V.

- (Hamkins) If $\mathbb{M}$ is a ground of V then $\mathbb{M}$ has no nontrivial grounds.

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- (Hamkins) $\mathbb{M} \models \mathrm{ZF}$ but must $\mathbb{M} \models \mathrm{ZFC}$?

The Directed Grounds Problem

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Claim

Suppose that grounds of V are downwards set-directed. Then the following are equivalent.

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Claim

Suppose that grounds of V are downwards set-directed and let $\mathbb{M}$ be the mantle of V. Then

$$\mathbb{M} \models \mathrm{ZFC}.$$

Bukovsky's Theorem and Usuba's Solution

Theorem (Bukovsky)

Suppose that κ is a regular cardinal and $N \subset V$ is an inner model. Then the following are equivalent.

1. *For each $\theta \in \mathrm{Ord}$ and for each function $F : \theta \to N$ there exists a function*
$$H : \theta \to \mathcal{P}_\kappa(N)$$
such that $H \in N$ and such that $F(\alpha) \in H(\alpha)$ for all $\alpha < \theta$.

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The grounds of V are downward set-directed under inclusion.

Corollary (Usuba)

Let $\mathbb{M}$ be the mantle of V.

- *Then $\mathbb{M} \models$ The Axiom of Choice.*

Usuba's Mantle Theorem

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Corollary

Suppose that there is an extendible cardinal. Let $\mathbb{M}$ be the mantle of V and suppose that $\mathbb{M} \subseteq \mathrm{HOD}$.

- *Then HOD is a ground of V.*

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Suppose that there is an extendible cardinal. Let $\mathbb{M}$ be the mantle of V and suppose that $\mathbb{M} \subseteq \mathrm{HOD}$.

- *Then HOD is a ground of V.*

- In this case, the **far** option in the HOD Dichotomy Theorem **cannot hold**.

A natural conjecture

Assuming sufficient large cardinals exist, then **provably** the far option in the HOD Dichotomy Theorem cannot hold.

The HOD Hypothesis

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- It is not known if there can exist 4 regular cardinals which **are** ω-strongly measurable in HOD.

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Conjecture

Suppose $\gamma > 2^{\aleph_0}$ and that γ^+ is ω-strongly measurable in HOD.

- Then γ^{++} is not ω-strongly measurable in HOD.

The HOD Conjecture

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The theory

$$\mathrm{ZFC} + \text{“There is a supercompact cardinal”}$$

proves the HOD Hypothesis.

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- The HOD Conjecture is a **number theoretic statement**.

The Weak HOD Conjecture and the $\mathrm{Ultimate}$-L Conjecture

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Ultimate-L Conjecture (weak version)

(ZFC) *Suppose that δ is an extendible cardinal. Then (provably) there is an inner model N such that:*

1. *N has the δ-approximation property and the δ-cover property.*
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Theorem

The Ultimate-L Conjecture implies the Weak HOD Conjecture.

An equivalence

Theorem

Suppose there is a proper class of extendible cardinals. Then following are equivalent.

(1) *The* HOD *Hypothesis holds.*

(2) *For some* δ*, there is an inner model* N *with the* δ*-approximation property and the* δ*-cover property such that*

$$N \models \text{“The HOD Hypothesis”}.$$

Weak extender models and universality

Definition

Suppose N is an inner model.

- Then N is a **weak extender model of δ is supercompact** if for every $\gamma > \delta$ there exists a normal fine δ-complete ultrafilter U on $\mathcal{P}_\delta(\gamma)$ such thta:
  - $N \cap \mathcal{P}_\delta(\gamma) \in U$,
  - $U \cap N \in N$.

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Universality Theorem (weak version)

Suppose N is a weak extender model of δ is supercompact and that U is a δ-complete ultrafilter on λ for some $\lambda \geq \delta$.

- *Then $U \cap N \in N$.*

The HOD Dichotomy (full version)

Theorem (HOD Dichotomy Theorem)

Suppose that δ is an extendible cardinal. Then one of the following holds.

(1) *No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD.

Further:

- HOD *is a weak extender model of δ is supercompact.*

The HOD Dichotomy (full version)

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Suppose that δ is an extendible cardinal. Then one of the following holds.

(1) *No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.*

Further:

- HOD *is a weak extender model of δ is supercompact.*

(2) *Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.*

Further:

- HOD *is not a weak extender of λ is supercompact, for any λ.*
- *There is no weak extender model N of λ is supercompact such that $N \subseteq \mathrm{HOD}$, for any λ.*

A unconditional corollary

Theorem

Suppose that δ is an extendible cardinal, $\kappa \geq \delta$, and that κ is a measurable cardinal.

- *Then κ is a measurable cardinal in HOD.*

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Two cases by appealing to the HOD Dichotomy Theorem:

- **Case 1:** HOD is **close** to V. Then HOD is a weak extender model of δ is supercompact.
  - Apply the Universality Theorem.

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Two cases by appealing to the HOD Dichotomy Theorem:

- **Case 1:** HOD is **close** to V. Then HOD is a weak extender model of δ is supercompact.
  - Apply the Universality Theorem.
- **Case 2:** HOD is **far** from V. Then **every** regular cardinal $\kappa \geq \delta$ is a measurable cardinal in HOD;
  - since κ is ω-strongly measurable in HOD.

Weak extender models, approximation, and cover

Theorem

Suppose N is a weak extender model of δ is supercompact.

- *Then N has the δ-approximation property and the δ-cover property.*

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Suppose N is a weak extender model of δ is supercompact. Thus:

- N is uniquely specified by $N \cap H(\delta^+)$.
- N is Σ_2-definable from $N \cap H(\delta^+)$.
    - The theory of weak extender models is part of the theory of V.

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Theorem

Suppose there is an extendible cardinal and that N is an inner model. Then the following are equivalent.

- *N has the δ-approximation property and the δ-cover property, for some δ.*
- *N is a weak extender model of δ is supercompact, for some δ.*

The δ-genericity property and strong universality

Definition

Suppose that N is an inner model and δ is an uncountable regular cardinal.

- Then N has the δ-**genericity property** if for all $\sigma \subset \delta$, if $|\sigma| < \delta$ then σ is N-generic for some partial $\mathbb{P} \in N$ such that $|\mathbb{P}| < \delta$.

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Suppose that δ is strongly inaccessible.

- Then HOD has the δ-genericity property.

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Suppose that δ is strongly inaccessible.

- Then HOD has the δ-genericity property.

Theorem

Suppose there is an extendible cardinal and that

- *N has the δ-approximation property, the δ-cover property, and the δ-genericity property.*

Suppose that the $\mathrm{Axiom\ I}_0$ *holds at λ, for some $\lambda > \delta$.*

- *Then in N, the* $\mathrm{Axiom\ I}_0$ *holds at λ, for some $\lambda > \delta$.*

A new family of inner models with the approximation and cover properties

Theorem

Suppose N is a weak extender model of δ is supercompact and that N has the δ-genericity property.
Suppose $U \in V_\delta$ is a countably complete ultrafilter and that

$$N_U = \mathrm{Ult}_0(N, U).$$

Then:

- *N_U has the δ-cover property.*
- *N_U has the δ-approximation property.*

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- *N_U has the δ-cover property.*
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- Assume δ is a strong cardinal and that N has the δ-approximation property and the δ-cover property.
  - N_U has the δ-cover property.
  - N_U can fail to have the δ-approximation property:
    - **Even** if $N = V$.

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Theorem

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- *Then there is no nontrivial elementary embedding*

  $$\pi : N \cap V_{\lambda+2} \to N \cap V_{\lambda+2}$$

  such that $\mathrm{CRT}(\pi) \geq \delta$.

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The following are equivalent.

1. *L is far from V (as in the Jensen Dichotomy Theorem).*
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Theorem

Suppose that δ is a supercompact cardinal.

- *Then there exists a weak extender model N of δ is supercompact such that*
  - *$N^\omega \subset N$.*
  - *There is a nontrivial elementary embedding $j : N \to N$.*

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Applications of the HOD Conjecture in ZF

Theorem (ZF)

Assume the HOD Conjecture and that there is a proper class of extendible cardinals.

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Then for every regular cardinal $\lambda \geq \delta$:

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- *The Solovay Splitting Theorem holds at λ.*

- Assuming the HOD Conjecture:
  - Large cardinal axioms are trying to prove the Axiom of Choice.

Berkeley cardinals

Definition

A cardinal δ is a **Berkeley cardinal** if:

- For all $\alpha < \delta$ and for all transitive sets M with $\delta \subset M$, there exists a nontrivial elementary embedding
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Theorem (ZF)

Assume the HOD *Conjecture. Then:*

- *There are no Berkeley cardinals.*

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There is a progression of theorems from large cardinal hypotheses that suggest:

- Some version of $V = L$ is true.

Further:

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Perhaps this is all evidence that $V = \mathrm{Ultimate}\text{-}L$.