

To infinity and far, far beyond

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Ordinals: the transfinite numbers

- ▶ \emptyset is the smallest ordinal: this is 0.
- ▶ $\{\emptyset\}$ is the next ordinal: this is 1.
- ▶ $\{\emptyset, \{\emptyset\}\}$ is next ordinal: this is 2.

If α is an ordinal then

- ▶ α is just the set of all ordinals β such that β is smaller than α ,
- ▶ $\alpha + 1 = \alpha \cup \{\alpha\}$ is the next largest ordinal.

ω denotes the least infinite ordinal, it is the set of all finite ordinals.

V: The Universe of Sets

The power set

Suppose X is a set. The **powerset** of X is the set

$$\mathcal{P}(X) = \{Y \mid Y \text{ is a subset of } X\}.$$

Cumulative Hierarchy of Sets

The universe V of sets is generated by defining V_α by induction on the ordinal α :

1. $V_0 = \emptyset$,
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
3. if α is a limit ordinal then $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$.

► If X is a set then $X \in V_\alpha$ for some ordinal α .

- ▶ $V_0 = \emptyset$, $V_1 = \{\emptyset\}$, $V_2 = \{\emptyset, \{\emptyset\}\}$.
 - ▶ These are just the ordinals: 0, 1, and 2.
- ▶ V_3 has 4 elements (and is clearly not an ordinal).
- ▶ V_4 has 16 elements.
- ▶ V_5 has 65,536 elements.
- ▶ V_{1000} has a lot of elements.

V_ω is infinite, it is the set of all (hereditarily) finite sets.

The conception of V_ω is **mathematically identical** to the conception of the structure $(\mathbb{N}, +, \cdot)$:

- ▶ Each structure can be interpreted in the other structure.

Beyond the basic axioms: large cardinal axioms

Shaping the conception of V

- ▶ The ZFC axioms of Set Theory **formally** specify the founding principles for the conception of V .
- ▶ The ZFC axioms are naturally augmented by additional axioms which assert the existence of “very large” infinite sets.
 - ▶ Such axioms assert the existence of **large cardinals**.

These large cardinals include:

- ▶ Measurable cardinals
- ▶ Strong cardinals
- ▶ Woodin cardinals
- ▶ Superstrong cardinals
- ▶ Supercompact cardinals
- ▶ Extendible cardinals
- ▶ Huge cardinals
- ▶ ω -huge cardinals

Cardinality: measuring the size of sets

Definition: when two sets have the same size

Two sets, X and Y , have the same **cardinality** if there is a matching of the elements of X with the elements of Y .

Formally: $|X| = |Y|$ if there is a bijection

$$f : X \rightarrow Y$$

Assuming the **Axiom of Choice** which is one of the ZFC axioms:

Theorem (Cantor)

For every set X there is an ordinal α such that $|X| = |\alpha|$.

The Continuum Hypothesis: CH

Theorem (Cantor)

The set \mathbb{N} of all natural numbers and the set \mathbb{R} of all real numbers do not have the same cardinality.

- ▶ There really are different “sizes” of infinity!

The Continuum Hypothesis

Suppose $A \subseteq \mathbb{R}$ is infinite. Then either:

1. A and \mathbb{N} have the same cardinality, or
2. A and \mathbb{R} have the same cardinality.

- ▶ This is Cantor's Continuum Hypothesis.

The Continuum Hypothesis and Mathematics

In 1900, at the second international mathematical congress in Paris, Hilbert posed his (now famous) list of 23 questions to guide mathematics for the “coming century”.

- ▶ The problem of the Continuum Hypothesis was the first problem on the list.

In 1904, at the third international mathematical congress in Heidelberg, König gave a lecture in which he claimed to solve CH.

- ▶ All parallel sessions of the congress were cancelled so that everyone could attend König's lecture.
- ▶ The announcement was a sensation and was widely reported by the press.
- ▶ The Grand Duke Friedrich I of Prussia had Felix Klein explain the entire matter to him personally.
- ▶ (The proof was wrong)

Many others tried to solve the problem of the Continuum Hypothesis and failed.

The problem of the Continuum Hypothesis quickly came to be widely regarded as one of the most important problems in all of modern Mathematics.

In 1940, Gödel showed that it is consistent with the axioms of Set Theory that the Continuum Hypothesis be true.

- ▶ One **cannot refute** the Continuum Hypothesis.

In 1963, on July 4th, Cohen announced in a lecture at Berkeley that it is consistent with the axioms of Set Theory that the Continuum Hypothesis be false.

- ▶ One **cannot prove** the Continuum Hypothesis.

Cohen's method

If M is a model of ZFC then M contains “blueprints” for virtual models N of ZFC, which enlarge M . These blueprints can be constructed and analyzed from within M .

- ▶ If M is countable then every blueprint constructed within M can be realized as genuine enlargement of M .
- ▶ Cohen proved that **every** model of ZFC contains a blueprint for an enlargement in which the Continuum Hypothesis is false.
- ▶ Cohen's method also shows that **every** model of ZFC contains a blueprint for an enlargement in which the Continuum Hypothesis is true.
- ▶ (Levy-Solovay) These enlargements preserve large cardinal axioms:
 - ▶ So if large cardinal axioms can help
 - ▶ it can only be in some **unexpected** way.

The extent of Cohen's method: It is not just about CH

A challenging time for the conception of V

- ▶ Cohen's method has been vastly developed in the 5 decades since Cohen's original work.
- ▶ Many problems have been showed to be unsolvable including problems outside Set Theory:
 - ▶ (Group Theory) Whitehead Problem (Shelah)
 - ▶ (Analysis) Kaplansky's Conjecture (Solovay)
 - ▶ (Combinatorics of the real line) Suslin's Problem (Solovay-Tennenbaum, Jensen, Jech)
 - ▶ (Measure Theory) Borel Conjecture (Laver)
 - ▶ (Operator Algebras) Brown-Douglas-Fillmore Automorphism Problem (Phillips-Weaver, Farah)
- ▶ This is a serious challenge to the very conception of Mathematical Infinity.
 - ▶ These examples, including the Continuum Hypothesis, are all statements about **just** $V_{\omega+2}$.

Ok, maybe it is just time to give up

Claim

- ▶ *Large cardinal axioms are not provable;*
 - ▶ by Gödel's Second Incompleteness Theorem.
- ▶ **But, large cardinal axioms are falsifiable.**

Prediction

No contradiction from the existence of infinitely many Woodin cardinals will be discovered within the next 1000 years.

- ▶ **Not by any means whatsoever.**

Truth beyond our formal reach

The real claim of course is:

- ▶ There is **no** contradiction from the existence of infinitely many Woodin cardinals.

Claim

- ▶ Such statements cannot be **formally** proved.
- ▶ This suggests there is a component in the evolution of our understanding of Mathematics which is **not** formal.
 - ▶ There is mathematical knowledge which is not entirely based in proofs.

Claim

The skeptical assessment that the conception of the universe of sets is incoherent, must be wrong.

- ▶ How else can these truths and ensuing predictions be explained?
- ▶ **But then either CH must be true or CH must be false.**

OK, back to the problem of the Continuum Hypothesis

The skeptic's challenge

Resolve the problem of CH.

- ▶ Perhaps one should begin by trying to more deeply understand CH.

A natural conjecture

One can more deeply understand CH by looking at special cases.

- ▶ But which special cases?
 - ▶ Does this even make sense?

The simplest uncountable sets

Definition

A set $A \subseteq V_{\omega+1}$ is a **projective set** if:

- ▶ A can be logically defined in the structure

$$(V_{\omega+1}, \in)$$

from parameters.

We can easily extend the definition to relations on $V_{\omega+1}$:

Definition

A set $A \subseteq V_{\omega+1} \times V_{\omega+1}$ is a **projective set** if:

- ▶ A can be logically defined as a binary relation in the structure

$$(V_{\omega+1}, \in)$$

from parameters.

- ▶ The countable subsets of $V_{\omega+1}$ and $V_{\omega+1} \times V_{\omega+1}$ are projective sets but so are $V_{\omega+1}$ and $V_{\omega+1} \times V_{\omega+1}$ themselves, and these sets are not countable.

The Continuum Hypothesis and the Projective Sets

The Continuum Hypothesis

Suppose $A \subseteq V_{\omega+1}$ is infinite. Then either:

1. A and V_{ω} have the same cardinality, or
2. A and $V_{\omega+1}$ have the same cardinality.

► This is a statement about **all** subsets of $V_{\omega+1}$.

The projective Continuum Hypothesis

Suppose $A \subseteq V_{\omega+1}$ is an infinite projective set. Then either:

1. A and V_{ω} have the same cardinality, or
2. There is a bijection

$$F : V_{\omega+1} \rightarrow A$$

such that F is a projective set.

► This is a statement about just the “simple” subsets of $V_{\omega+1}$.

The Axiom of Choice

Definition

Suppose that

$$A \subseteq X \times Y$$

A function

$$F : X \rightarrow Y$$

is a **choice function** for A if for all $a \in X$:

- ▶ If there exists $b \in Y$ such that $(a, b) \in A$ then $(a, F(a)) \in A$.

The Axiom of Choice

For every set

$$A \subseteq X \times Y$$

there exists a choice function for A .

The Axiom of Choice and the Projective Sets

The projective Axiom of Choice

Suppose $A \subseteq V_{\omega+1} \times V_{\omega+1}$ is a projective set. Then there is a function

$$F : V_{\omega+1} \rightarrow V_{\omega+1}$$

such that:

- ▶ F is a choice function for A .
 - ▶ F is a projective set.
-
- ▶ There were many attempts in the early 1900s to solve both the problem of projective Continuum Hypothesis and the problem of the projective Axiom of Choice:
 - ▶ Achieving success for the simplest instances.
 - ▶ However, by 1925 these problems both looked hopeless.

These were both hopeless problems

The actual constructions of Gödel and Cohen show that both problems are formally unsolvable.

- ▶ In Gödel's universe L :
 - ▶ The projective Axiom of Choice holds.
 - ▶ The projective Continuum Hypothesis holds.
- ▶ In the Cohen enlargement of L (as given by the actual blueprint which Cohen defined for the failure of CH):
 - ▶ The projective Axiom of Choice is false.
 - ▶ The projective Continuum Hypothesis is false.
- ▶ This explains why these problems were so difficult.
- ▶ But the intuition that these problems **are solvable** was correct.

An unexpected entanglement

Theorem (1984)

Suppose there are infinitely many Woodin cardinals. Then:

- ▶ *The projective Continuum Hypothesis holds.*

Theorem (1985: Martin-Steel)

Suppose there are infinitely many Woodin cardinals. Then:

- ▶ *The projective Axiom of Choice holds.*

We now have the correct conception of $V_{\omega+1}$ and the projective sets.

- ▶ This conception yields axioms for the projective sets.
- ▶ These (determinacy) axioms in turn are closely related to (and follow from) large cardinal axioms.

But what about $V_{\omega+2}$? Or even V itself?

Logical definability

The definable power set

For each set X , $\mathcal{P}_{\text{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that Y is logically definable in the structure (X, \in) from parameters in X .

- ▶ $\mathcal{P}_{\text{Def}}(X)$ is the collection of just those subsets of X which are intrinsic to X itself,
 - ▶ versus $\mathcal{P}(X)$ which is the collection of **all** subsets of X .

The collection of all the projective subsets of $V_{\omega+1}$ is exactly given by:

$$\mathcal{P}_{\text{Def}}(V_{\omega+1})$$

The effective cumulative hierarchy: L

Cumulative Hierarchy of Sets

The cumulative hierarchy is defined by induction on α as follows.

1. $V_0 = \emptyset$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. if α is a limit ordinal then $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$.

► V is the class of all sets X such that $X \in V_\alpha$ for some α .

Gödel's constructible universe, L

Define L_α by induction on α as follows.

1. $L_0 = \emptyset$.
2. $L_{\alpha+1} = \mathcal{P}_{\text{Def}}(L_\alpha)$.
3. if α is a limit ordinal then $L_\alpha = \bigcup \{L_\beta \mid \beta < \alpha\}$.

► L is the class of all sets X such that $X \in L_\alpha$ for some α .

The missing axiom for V ?

The axiom: $V = L$

Suppose X is a set. Then $X \in L$.

Theorem (Gödel:1940)

Assume $V = L$. Then the Continuum Hypothesis holds.

- ▶ Suppose there is a Cohen-blueprint for $V = L$. Then:
 - ▶ the axiom $V = L$ must hold and the blueprint is trivial.

Claim

Adopting the axiom $V = L$ completely negates the ramifications of Cohen's method.

- ▶ Could this be the resolution?
- ▶ No, there is a **serious** problem.

The axiom $V = L$ and large cardinals

Theorem (Scott:1961)

Assume $V = L$. Then there are no measurable cardinals.

▶ *In fact there are no (genuine) large cardinals.*

▶ Assume $V = L$. **Then there are no Woodin cardinals.**

Clearly:

The axiom $V = L$ is false.

A natural conjecture

Perhaps the key is to generalize the construction of L by using the large cardinals to **expand** the definable powerset operation.

▶ But there is an alternative approach which is based on simply using the large cardinals to directly generalize the projective sets.

Another way to define the projective sets

Observation

$V_{\omega+1}$ is homeomorphic to the Cantor set, with the topology on $V_{\omega+1}$ given by the sets

$$O_{n,a} = \{X \subseteq V_{\omega} \mid X \cap V_n = a\}$$

as basic open sets where $n < \omega$ and $a \in V_{n+1}$.

- ▶ The projective subsets of $V_{\omega+1}$ are exactly the sets generated from the open sets and closing under the operations:
 - ▶ Taking images by continuous functions
$$F : V_{\omega+1} \rightarrow V_{\omega+1}.$$
 - ▶ Taking complements.

- ▶ This definition generalizes to any topological space.
- ▶ In particular, this extends the notion of the projective sets to the Euclidean spaces \mathbb{R}^n .

Universally Baire sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is **universally Baire** if:

- ▶ For all topological spaces Ω
- ▶ For all continuous functions $\pi : \Omega \rightarrow \mathbb{R}^n$;

the preimage of A by π has the property of Baire in the space Ω .

- ▶ Universally Baire sets have the property of Baire
 - ▶ Simply take $\Omega = \mathbb{R}^n$ and π to be the identity.
- ▶ Universally Baire sets are Lebesgue measurable.

Theorem

Assume $V = L$. Then every set $A \subseteq \mathbb{R}$ is the image of a universally Baire set by a continuous function

$$F : \mathbb{R} \rightarrow \mathbb{R}.$$

$L(A, \mathbb{R})$ where $A \subseteq \mathbb{R}$

Relativizing L to $A \subseteq \mathbb{R}$

Suppose $A \subseteq \mathbb{R}$. Define $L_\alpha(A, \mathbb{R})$ by induction on α by:

1. $L_0(A, \mathbb{R}) = V_{\omega+1} \cup \{A\}$,
2. (Successor case) $L_{\alpha+1}(A, \mathbb{R}) = \mathcal{P}_{\text{Def}}(L_\alpha(A, \mathbb{R}))$,
3. (Limit case) $L_\alpha(A, \mathbb{R}) = \cup\{L_\beta(A, \mathbb{R}) \mid \beta < \alpha\}$.

- ▶ $L(A, \mathbb{R})$ is the class of all sets X such that $X \in L_\alpha(A, \mathbb{R})$ for some ordinal α .

- ▶ $\mathcal{P}(\mathbb{R}) \cap L_{\omega_1}(A, \mathbb{R})$ is the smallest σ -algebra containing A and closed under images by continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$.
 - ▶ If $B \subseteq \mathbb{R}$ and $B \in L(A, \mathbb{R})$ then $L(B, \mathbb{R}) \subseteq L(A, \mathbb{R})$. So:
 - ▶ $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is closed under images by continuous functions

$$F : \mathbb{R} \rightarrow \mathbb{R}.$$

The universally Baire sets are the ultimate generalization of the projective sets

Theorem

Suppose that there is a proper class of Woodin cardinals and suppose $A \subseteq \mathbb{R}$ is universally Baire.

- ▶ *Then every set $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is universally Baire.*
- ▶ Thus every projective set is universally Baire.
 - ▶ Since clearly there exists a proper class of Woodin cardinals.

Theorem

Suppose that there is a proper class of Woodin cardinals.

- (1) (Martin-Steel) *Suppose $A \subseteq \mathbb{R}$ is universally Baire.*
 - ▶ *Then A is determined.*
- (2) (Steel) *Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is universally Baire.*
 - ▶ *Then A has a choice function which is universally Baire.*

- ▶ Thus $L(A, \mathbb{R}) \models \text{AD}$, where AD is the Axiom of Determinacy.

Measuring the complexity of universally Baire sets

Definition

Suppose A and B are subsets of \mathbb{R} .

1. A is **weakly Wadge reducible** to B , $A \leq_{\text{Wadge}} B$, if there is a function $\pi : \mathbb{R} \rightarrow \mathbb{R}$ such that:
 - ▶ π is continuous on $\mathbb{R} \setminus \mathbb{Q}$.
 - ▶ Either $A = \pi^{-1}[B]$ or $A = \mathbb{R} \setminus \pi^{-1}[B]$.
 2. A and B are **weakly Wadge bi-reducible** if
 - ▶ $A \leq_{\text{Wadge}} B$ and $B \leq_{\text{Wadge}} A$.
 3. The **weak Wadge degree** of A is the equivalence class of all sets which are weakly Wadge bi-reducible with A .
- ▶ If A is weakly Wadge reducible to B and B is universally Baire then A is universally Baire.

An indication of deep structure

Theorem (Martin-Steel, Martin, Wadge)

Assume there is a proper class of Woodin cardinals.

Then the weak Wadge degrees of the universally Baire sets are linearly ordered by weak Wadge reducibility and moreover this is a wellorder.

Speculation

Perhaps this ultimate generalization of the projective sets can lead us to the ultimate generalization of the axiom $V = L$.

- ▶ But how?

Transitive sets and Σ_2 -sentences

Definition

A set M is a **transitive set** if $a \subset M$ for each $a \in M$.

- ▶ Each set V_α is a transitive set.
- ▶ Suppose M is a finite transitive set.
 - ▶ Then $M \in V_\omega$.
- ▶ V_ω is the union of all finite transitive sets.

A sentence φ is a Σ_2 -sentence if it is of the form:

- ▶ *There exists an ordinal α such that $V_\alpha \models \psi$;
for some sentence ψ .*

Defining the axiom $V = L$ without defining L

For each ordinal α , let N_α be the intersection of all transitive sets M such that

- ▶ $M \models \text{ZFC} \setminus \text{Powerset}$,
- ▶ $\text{Ord}^M = \alpha$.

Lemma

The following are equivalent.

- (1) $V = L$.
- (2) *For each Σ_2 -sentence φ , if $V \models \varphi$ then there exists a countable ordinal α such that $N_\alpha \models \varphi$.*

- ▶ We need to somehow use the universal Baire sets in a reformulation of (2).

Gödel's transitive class HOD

Definition

HOD is the class of all sets X such that there exist $\alpha \in \text{Ord}$ and $M \in V_\alpha$ such that

1. $X \in M$ and M is transitive.
2. Every element of M is definable in V_α from ordinal parameters.

- For every set b there is a minimum transitive set $\text{TC}(b)$ which contains b as an element.

Why HOD?

*Suppose N is a model of ZF. Let $\text{HOD}^N \subseteq N$ be HOD as defined **within** N . Then for each $b \in N$, the following are equivalent:*

1. $b \in \text{HOD}^N$.
2. *Every element of $(\text{TC}(b))^N$ is definable in N with parameters from the ordinals of N .*

$\text{HOD}^{L(A, \mathbb{R})}$ and measurable cardinals

Definition

Suppose that $A \subseteq \mathbb{R}$. Then $\text{HOD}^{L(A, \mathbb{R})}$ is the class HOD as defined within $L(A, \mathbb{R})$.

- ▶ The Axiom of Choice must hold in $\text{HOD}^{L(A, \mathbb{R})}$
 - ▶ even if $L(A, \mathbb{R}) \models \text{AD}$.

Theorem (Solovay:1967)

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \text{AD}$.

- ▶ *Then ω_1^V is a measurable cardinal in $\text{HOD}^{L(A, \mathbb{R})}$.*
- ▶ Solovay's theorem gave the first connection between the Axiom of Determinacy (AD) and large cardinal axioms.

The least measurable cardinal of $\text{HOD}^{L(A, \mathbb{R})}$

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

- ▶ *Then ω_1^V is the least measurable cardinal in $\text{HOD}^{L(A, \mathbb{R})}$.*
- ▶ If stronger large cardinals exist in $\text{HOD}^{L(A, \mathbb{R})}$, they can only occur above ω_1^V .

$\text{HOD}^{L(A, \mathbb{R})}$ and Woodin cardinals

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Then

- ▶ $\Theta^{L(A, \mathbb{R})}$ is the supremum of the ordinals α such that there exists a surjection, $\pi : \mathbb{R} \rightarrow \alpha$, such that $\pi \in L(A, \mathbb{R})$.

Lemma

Suppose that $A \subseteq \mathbb{R}$. Then:

- ▶ *There are no measurable cardinals in $\text{HOD}^{L(A, \mathbb{R})}$ above $\Theta^{L(A, \mathbb{R})}$.*

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire. Then:

- ▶ $\Theta^{L(A, \mathbb{R})}$ is a Woodin cardinal in $\text{HOD}^{L(A, \mathbb{R})}$.

The axiom $V = \text{Ultimate-L}$

- ▶ The existence of a Woodin cardinal is expressible by a Σ_2 -sentence.
 - ▶ Woodin cardinals clearly exist in V ;
 - ▶ If $A \subseteq \mathbb{R}$ is universally Baire and there is a proper class of Woodin cardinals then

$$\text{HOD}^{L(A, \mathbb{R})} \models \text{“There is a Woodin cardinal”}.$$

The axiom for $V = \text{Ultimate-L}$

- ▶ *There is a proper class of Woodin cardinals.*
- ▶ *For each Σ_2 -sentence φ , if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that*

$$\text{HOD}^{L(A, \mathbb{R})} \models \varphi.$$

This is just rank resemblance

Assume there is a proper class of Woodin cardinals. Then the following are equivalent:

- ▶ $V = \text{Ultimate-}L$.
- ▶ Suppose ψ is a sentence and there exists an ordinal α such that

$$V_\alpha \models \psi.$$

- ▶ Then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\text{HOD}^{L(A, \mathbb{R})} \models \text{“There exists } \alpha \text{ such that } V_\alpha \models \psi\text{”}$$

Some consequences of $V = \text{Ultimate-}L$

Theorem ($V = \text{Ultimate-}L$)

The Continuum Hypothesis holds.

Theorem ($V = \text{Ultimate-}L$)

$V = \text{HOD}$.

The axiom $V = \text{Ultimate-}L$ and Cohen's method

- ▶ Suppose there is a Cohen-blueprint for $V = \text{Ultimate-}L$.
Then:
 - ▶ The axiom $V = \text{Ultimate-}L$ must hold and the blueprint is trivial.
- ▶ The axiom $V = \text{Ultimate-}L$ settles (modulo axioms of infinity) *all* sentences about “small” sets (like $V_{\omega+2}$) which have been shown to be independent by Cohen's method.

Claim

Adopting the axiom $V = \text{Ultimate-}L$ completely negates the ramifications of Cohen's method.

- ▶ But, is the axiom $V = \text{Ultimate-}L$ compatible with all large cardinal axioms?
 - ▶ Is there a Scott Theorem for $V = \text{Ultimate-}L$?

The language of large cardinals: elementary embeddings

Definition

Suppose X and Y are transitive sets. A function $j : X \rightarrow Y$ is an **elementary embedding** if for all logical formulas $\varphi[x_0, \dots, x_n]$ and all $a_0, \dots, a_n \in X$,

$$(X, \in) \models \varphi[a_0, \dots, a_n] \text{ if and only if } (Y, \in) \models \varphi[j(a_0), \dots, j(a_n)]$$

- ▶ Isomorphisms are elementary embeddings but the only isomorphisms of (X, \in) and (Y, \in) are trivial.

Extendible cardinals

Lemma

Suppose that $j : V_\alpha \rightarrow V_\beta$ is an elementary embedding. Then the following are equivalent.

- (1) j is not the identity.
- (2) There is an ordinal $\eta < \alpha$ such that $j(\eta) \neq \eta$.

► $\text{CRT}(j)$ denotes the least ordinal η such that $j(\eta) \neq \eta$.

Definition (Reinhardt:(1974))

Suppose that δ is a cardinal.

- Then δ is an **extendible cardinal** if for each $\lambda > \delta$ there exists an elementary embedding

$$j : V_{\lambda+1} \rightarrow V_{j(\lambda)+1}$$

such that $\text{CRT}(j) = \delta$ and $j(\delta) > \lambda$.

The transitive sets $H(\gamma)$

Definition

For each (infinite) cardinal γ :

- ▶ $H(\gamma)$ denotes the union of all transitive sets M such that $|M| < \gamma$.
- ▶ $H(\omega) = V_\omega$.
- ▶ $H(\omega_1)$ and $V_{\omega+1}$ are logically equivalent structures.
- ▶ $H(\omega_1) \models \text{ZFC} \setminus \text{Powerset}$.
- ▶ For any infinite cardinal γ , $H(\gamma^+) \models \text{ZFC} \setminus \text{Powerset}$.

Inner models

A transitive class is an **inner model** if

- ▶ $\text{Ord} \subset M$,
- ▶ $M \models \text{ZFC}$.

(meta) Lemma

Suppose M is a transitive class containing Ord . Then the following are equivalent:

- ▶ *M is an inner model.*
- ▶ *For each infinite cardinal γ ,*

$$M \cap H(\gamma^+) \models \text{ZFC} \setminus \text{Powerset}.$$

The δ -cover and δ -approximation properties

Definition (Hamkins)

Suppose N is an inner model and that δ is an uncountable (regular) cardinal of V .

1. N has the δ -**covering property** if for all $\sigma \in N$, if $|\sigma| < \delta$ then there exists $\tau \in N$ such that:
 - ▶ $\sigma \subset \tau$,
 - ▶ $\tau \in N$,
 - ▶ $|\tau| < \delta$.
 2. N has the δ -**approximation property** if for all sets $X \in N$, the following are equivalent.
 - ▶ $X \in N$.
 - ▶ For all $\sigma \in N$ if $|\sigma| < \delta$ then $\sigma \cap X \in N$.
- ▶ If V is an extension of an inner model N by Cohen's method then for all sufficiently large regular cardinals δ :
- ▶ N has the δ -approximation property.
 - ▶ N has the δ -cover property.

The Hamkins Uniqueness Theorem

Theorem (Hamkins)

Suppose N_0 and N_1 both have the δ -approximation property and the δ -cover property. Suppose

- ▶ $N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+).$

Then:

- ▶ $N_0 = N_1.$

- ▶ N is a Σ_2 -definable class from parameters.

Claim

The theory of inner models N of V which have the δ -approximation property and the δ -cover property, for some δ , is part of the (first-order) theory of V .

Large cardinals above δ are downward absolute to inner models with the δ -cover and δ -approximation properties

Theorem (Hamkins)

Suppose that N is an inner model with the δ -cover and δ -approximation properties, $\kappa > \delta$, and that κ is an extendible cardinal.

- ▶ *Then κ is an extendible cardinal in N .*
- ▶ There are generalizations of this for all large cardinal notions.
 - ▶ These are the universality theorems.
- ▶ **Conclusion:** There can be **no generalization** of Scott's Theorem to **any** axiom which holds in **some** inner model with the δ -cover and δ -approximation properties, for any δ .

The Ultimate- L Conjecture

Ultimate- L Conjecture (weak version)

(ZFC) *Suppose that δ is an extendible cardinal. Then (provably) there is an inner model N such that:*

1. N has the δ -cover and δ -approximation properties.
2. $N \models "V = \text{Ultimate-}L"$.

- ▶ The Ultimate- L Conjecture implies there is no generalization of Scott's theorem to the axiom $V = \text{Ultimate-}L$.
 - ▶ By the universality theorems.
- ▶ The Ultimate- L Conjecture is an existential number theoretic statement.
 - ▶ If it is undecidable then it must be **false**.

Claim

*The Ultimate- L Conjecture **must** be either true or false*

- ▶ *it cannot be meaningless.*

Set Theory faces one of two futures

- ▶ The Ultimate- L Conjecture reduces the **entire** post-Cohen debate on Set Theoretic truth to a single question which
 - ▶ **must** have an answer.

Future 1: The Ultimate- L Conjecture is true.

- ▶ Then the axiom $V = \text{Ultimate-}L$ is very likely the key missing axiom for V .
 - ▶ There is **no** generalization of Scott's Theorem for the axiom $V = \text{Ultimate-}L$.
 - ▶ **All** the questions which have been shown to be unsolvable by Cohen's method are resolved modulo large cardinal axioms.

Future 2: The Ultimate- L Conjecture is false.

- ▶ Then the program to understand V by generalizing the success in understanding $V_{\omega+1}$ and the projective sets fails.

- ▶ Which is it?