Counting modulo \( N \) in pseudo-finite fields

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Outline

1. Model-theoretic Euler characteristics
2. Pseudo-finite fields and difference fields
3. \( \mathbb{Z}/N \)-valued Euler characteristics on pseudo-finite fields.
If $M$ is a structure, $\varphi(\vec{x}; \vec{y})$ is a first-order formula, and $\vec{m}$ is a tuple from $M$ with $|\vec{m}| = |\vec{y}|$, then

$$\varphi(M; \vec{m}) := \{ \vec{a} \in M^{\vec{x}} : M \models (\vec{a}; \vec{m}) \}$$

Sets of the form $\varphi(M; \vec{m})$ are called **definable sets**.
If $M$ is a structure, $\varphi(\vec{x}; \vec{y})$ is a first-order formula, and $\vec{m}$ is a tuple from $M$ with $|\vec{m}| = |\vec{y}|$, then

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Sets of the form $\varphi(M; \vec{m})$ are called **definable sets**.

**Definition**

If $X$ and $Y$ are definable sets, a function $f : X \rightarrow Y$ is a **definable function** if its graph $\Gamma_f \subseteq X \times Y$ is a definable set.
Definable families

Definition

If $X$ and $Y$ are definable sets, a family $\{D_b : b \in Y\}$ of subsets of $X$ is a **definable family** if the set

$$\bigsqcup_{b \in Y} D_b := \{(a, b) \in X \times Y : a \in D_b\}$$

is a definable subset of $X \times Y$. The definable set $\bigsqcup_{b \in Y} D_b$ is called the **total space** of the family.
Definition

A function \( f : X \to R \) from a definable set \( X \) to an abstract set \( R \) is **definable** if the range \( f(X) \) is finite and if \( f^{-1}(r) \) is a definable subset of \( X \) for every \( r \in R \).
Definable functions to abstract sets

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A function \( f : X \to R \) from a definable set \( X \) to an abstract set \( R \) is \textit{definable} if the range \( f(X) \) is finite and if \( f^{-1}(r) \) is a definable subset of \( X \) for every \( r \in R \).

Example
In an \( \aleph_1 \)-categorical theory, “Morley rank is definable” means that for every definable family \( \{D_a : a \in X\} \), the function

\[
a \mapsto RM(D_a)
\]

is a definable function from \( X \) to \( \mathbb{N} \).
O-minimality

**Definition**

An ordered structure \((M, <, \ldots)\) is *o-minimal* if every definable subset of \(M^1\) is a finite union of points and intervals.

Examples:

- The totally ordered set \((\mathbb{Q}, <)\)
- The ordered field \((\mathbb{R}, <, +, \cdot)\)

Definable sets are very well-behaved in o-minimal structures, thanks to dimension theory, cell decomposition, etc.
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Definable sets are very well-behaved in o-minimal structures, thanks to dimension theory, cell decomposition, etc.

**Definition**

An o-minimal structure \((M, <, \ldots)\) is \textit{dense} if the underlying order is dense, i.e.,

\[
M \models \forall x, y : (x < y) \implies (\exists z : (x < z) \land (z < y))
\]
Let $(M, <, \ldots)$ be a dense o-minimal structure. If $X$ is a definable set, then $X$ can be written as a disjoint union of cells

$$X = C_1 \bigsqcup C_2 \bigsqcup \cdots \bigsqcup C_n$$

Definition

The Euler characteristic of $X$ is

$$\chi(X) = \sum_{i=1}^{n} (-1)^{\dim(C_i)}$$

Proposition (van den Dries)

This doesn't depend on the choice of the cell decomposition.
O-minimal Euler characteristic

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Each cell \(C_i\) has a dimension \(\dim(C_i)\). If \(M = \mathbb{R}\), then cells of dimension \(k\) are homeomorphic to the \(k\)-dimensional open ball.
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O-minimal Euler characteristic satisfies the following additional properties:

1. If \( \{D_x : x \in X\} \) is a definable family, then the function

\[
x \mapsto \chi(D_x)
\]

is a definable function from \( X \) to \( \mathbb{Z} \).

Example

If \( 1 \to G_1 \to G_2 \to G_3 \to 1 \) is a short exact sequence of definable groups, then

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\chi(G_2) = \chi(G_1) \cdot \chi(G_3)
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Euler characteristic in topology

If $X$ is a sufficiently nice topological space, the Euler characteristic $\chi(X)$ is defined as the alternating sum of Betti numbers:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(X; \mathbb{C}).$$

When $X$ is a finite simplicial complex, $\chi(X)$ is given by a more familiar formula $\chi(X) = \sum_{i=0}^{\infty} (-1)^i F_i$ where $F_i$ is the number of $i$-dimensional simplices.

Example: Euler's formula $V - E + F = 2$ is the statement that the Euler characteristic of the sphere is 2.
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Example

Euler’s formula

$$V - E + F = 2$$

is the statement that the Euler characteristic of the sphere is 2.
The reason for the name ”Euler characteristic”

**Theorem (van den Dries)**

If $R$ is an o-minimal expansion of the real field $(\mathbb{R}, <, +, \cdot)$, and $X \subseteq \mathbb{R}^n$ is a definable and compact set, then the o-minimal Euler characteristic of $X$ agrees with the topological Euler characteristic.
Let $M$ be a structure and $R$ be a (commutative) ring.

### Definition

An $R$-valued Euler characteristic on $M$ is a map

$$\chi : \{\text{Definable sets in } M\} \rightarrow R$$

satisfying the axioms

1. $\chi(X) = \chi(Y)$ if there is a definable bijection $f : X \rightarrow Y$.
2. $\chi(\emptyset) = 0$
3. $\chi(X \cup Y) = \chi(X) + \chi(Y)$ if $X \cap Y = \emptyset$
4. $\chi(\{a\}) = 1$
5. $\chi(X \times Y) = \chi(X) \times \chi(Y)$
An Euler characteristic $\chi$ is \emph{definable} if for every definable family $\{D_x : x \in X\}$, the following function is definable:

$$x \mapsto \chi(D_x)$$
Strong and definable Euler characteristics

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Definition
An Euler characteristic $\chi$ is strong if for every definable family $\{D_x : x \in X\}$, if $\chi(D_x) = \alpha$ for all $x \in X$ then

$$ \chi\left(\bigsqcup_{x \in X} D_x\right) = \alpha \cdot \chi(X) $$
Examples

- O-minimal Euler characteristic
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- Size, in finite structures:
  \[ \chi(X) = \#X \]
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- O-minimal Euler characteristic
- Size, in finite structures:

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Both these examples are strong and definable.
An \( R \)-valued “measure” on a definable set \( X \) is a function

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\mu : \{ \text{Definable subsets of } X \} \to R
\]
satisfying the axioms:

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A measure $\mu$ is \textit{definable} if for any definable family $\{D_y : y \in Y\}$ of subsets of $X$, the following is a definable function from $Y$ to $R$:
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**Remark**

An $R$-valued Euler characteristic $\chi$ induces an $R$-valued measure $\chi|_X$ on every definable set $X$. The Euler characteristic $\chi$ is definable iff $\chi|_X$ is definable for every $X$. 
If $\mu$ is an $R$-valued measure on $X$, and $f : X \to R$ is definable, there is a well-defined “integral”

$$\int_{x \in X} f(x) \, d\mu(x).$$

Remark: This does not depend on the choice of the partition.
Integration against a “measure”

If $\mu$ is an $R$-valued measure on $X$, and $f : X \to R$ is definable, there is a well-defined “integral”

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This can be defined by partitioning $X$ into subsets

$$X = X_1 \bigsqcup \cdots \bigsqcup X_n$$
on which $f$ is constant

$$f(X_i) = \alpha_i,$$
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and then setting

$$\int_{x \in X} f(x) \, d\mu(x) := \sum_{i=1}^{n} \alpha_i \cdot \mu(X_i)$$

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Integration against a “measure”

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Remark

This does not depend on the choice of the partition.
Integration is $R$-linear:

$$\int_{x \in X} (a \cdot f(x) + g(x)) \, d\mu(x) = a \cdot \int_{x \in X} f(x) \, d\mu(x) + \int_{x \in X} g(x) \, d\mu(x).$$
Properties of integration

Integration is $\mathbb{R}$-linear:

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If $1_D$ is the characteristic function of a definable subset $D \subseteq X$, then

$$\int_{x \in X} 1_D(x) \, d\mu(x) = \mu(D).$$
Properties of integration

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If $1_D$ is the characteristic function of a definable subset $D \subseteq X$, then

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If $\mu$ is definable and $f : X \times Y \to R$ is definable, then

$$\int_{x \in X} f(x, y) \ d\mu(x)$$

is a definable function $Y \to R$. Moreover, this property characterizes definable measures.
Let $\chi$ be a strong Euler characteristic and $\{D_x : x \in X\}$ be a definable family. Suppose the following function $X \to R$ is definable:

$$x \mapsto \chi(D_x)$$

Then, for $\mu = \chi|_X$, we have

$$\chi(\biguplus x \in X D_x) = \int_{x \in X} \chi(D_x) \, d\mu(x)$$

In particular, $\chi$ of the total space is determined by $\chi$ of the fibers and the restriction of $\chi$ on the base.
Let $\chi$ be a strong Euler characteristic and \( \{D_x : x \in X\} \) be a definable family.

Suppose the following function $X \rightarrow \mathbb{R}$ is definable:

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In particular, \( \chi \) of the total space is determined by \( \chi \) of the fibers and the restriction of \( \chi \) on the base.
Let $\chi$ be a strong Euler characteristic on a 1-sorted structure $M$.

**Proposition**

The following are equivalent:

- $\chi$ is definable
- $\chi|_{M^1}$ is definable

Proof.

Let $\mu = \chi|_{M^1}$. Assuming $\mu$ is definable, one proves by induction that for any definable $X \subseteq M^n$,

$$
\chi(D) = \int_{x_1 \in M} \cdots \int_{x_n \in M} D(\vec{x}) \, d\mu(x_n) \cdots d\mu(x_1)
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Strong definable Euler characteristics

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\chi(D) = \int_{x_1 \in M} \cdots \int_{x_n \in M} 1_D(\vec{x}) \ d\mu(x_n) \cdots d\mu(x_1)
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Proposition

Let $M$ be a $1$-sorted structure and $\mu$ be a definable measure on $M^1$. Then $\mu$ comes from a strong definable Euler characteristic iff these axioms hold:

1. $\mu(\{a\}) = 1$ for every $a \in M^1$.
2. For every definable $D \subseteq M^2$, the Fubini property holds:

$$\int_{x \in M^1} \int_{y \in M^1} D(x, y) \, d\mu(y) \, d\mu(x) = \int_{y \in M^1} \int_{x \in M^1} D(x, y) \, d\mu(x) \, d\mu(y)$$

This can be used to construct the o-minimal Euler characteristic.
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This can be used to construct the o-minimal Euler characteristic.
Example: $ACF_0$

- The field $\mathbb{C}$ can be interpreted in the field $\mathbb{R}$. The o-minimal Euler characteristic $\chi_\mathbb{R}$ on $(\mathbb{R}, +, \cdot)$ yields a strong Euler characteristic $\chi_\mathbb{C}$ on $(\mathbb{C}, +, \cdot)$.
Example: \( ACF_0 \)

- The field \( \mathbb{C} \) can be interpreted in the field \( \mathbb{R} \). The o-minimal Euler characteristic \( \chi_\mathbb{R} \) on \((\mathbb{R}, +, \cdot)\) yields a strong Euler characteristic \( \chi_\mathbb{C} \) on \((\mathbb{C}, +, \cdot)\).
- The induced measure on \( \mathbb{C}^1 \) is

\[
\mu(X) = |X| \quad \text{if } X \text{ finite}
\]

\[
\mu(\mathbb{C}) = 1 \quad \text{(because } \mathbb{C} \text{ is a 2-cell)}
\]

\[
\mu(\mathbb{C} \setminus X) = 1 - |X| \quad \text{if } X \text{ is finite}
\]

This measure is definable, so \( \chi_\mathbb{C} \) is definable.

If \( K \models ACF_0 \), then the same \( \mu \) is a definable measure, and

\[
d\mu(x) = d\mu(y) = d\mu(x)
\]

because \( K \equiv \mathbb{C} \).

Therefore, \( \mu \) gives rise to a \( \mathbb{Z} \)-valued definable strong Euler characteristic on \( K \).
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- This measure is definable, so $\chi_{\mathbb{C}}$ is definable.
- If $K \models ACF_0$, then the same $\mu$ is a definable measure, and
  
  $$
  d\mu(x) \ d\mu(y) = d\mu(y) \ d\mu(x)
  $$

  because $K \equiv \mathbb{C}$. 
Example: $ACF_0$

- The field $\mathbb{C}$ can be interpreted in the field $\mathbb{R}$. The o-minimal Euler characteristic $\chi_\mathbb{R}$ on $(\mathbb{R}, +, \cdot)$ yields a strong Euler characteristic $\chi_\mathbb{C}$ on $(\mathbb{C}, +, \cdot)$.

- The induced measure on $\mathbb{C}^1$ is
  
  $$\mu(X) = |X| \quad \text{if } X \text{ finite}$$
  
  $$\mu(\mathbb{C}) = 1 \quad \text{(because } \mathbb{C} \text{ is a 2-cell)}$$
  
  $$\mu(\mathbb{C} \setminus X) = 1 - |X| \quad \text{if } X \text{ is finite}$$

- This measure is definable, so $\chi_\mathbb{C}$ is definable.

- If $K \models ACF_0$, then the same $\mu$ is a definable measure, and
  
  $$d\mu(x) \, d\mu(y) = d\mu(y) \, d\mu(x)$$

  because $K \equiv \mathbb{C}$.

- Therefore, $\mu$ gives rise to a $\mathbb{Z}$-valued definable strong Euler characteristic on $K$. 
A (commutative) semiring is a structure \((R, +, 0, 1, \cdot)\) such that

- \((R, 0, +)\) is a commutative monoid.
- \((R, 1, \cdot)\) is a commutative monoid.
- For every \(a \in R\), the map \(x \mapsto x \cdot a\) is a monoid endomorphism of \((R, 0, +)\).
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**Example**

The tropical semiring is $R = \{0, 1, \omega, \omega^2, \omega^3, \ldots\}$, where, for example

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\omega^2 + \omega^3 = \omega^3 \\
\omega^2 \cdot \omega^3 = \omega^5
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**Remark**

Semiring-valued Euler characteristics work just as well as ring-valued Euler characteristics!
Definition

A theory $T$ is geometric if

1. $\exists^\infty$ can be eliminated
2. acl satisfies Steinitz exchange:

$$a \in acl(\{b\} \cup S) \land a \notin acl(S) \implies b \in acl(\{a\} \cup S)$$
Definition

A theory $T$ is \textit{geometric} if

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\[ a \in \text{acl}(\{b\} \cup S) \land a \notin \text{acl}(S) \implies b \in \text{acl}(\{a\} \cup S) \]

Examples include ACVF, $\mathbb{Q}_p$, pseudo-finite fields, strongly minimal theories, o-minimal theories...
Remark

If $M$ is a model of a geometric theory, there is an Euler characteristic taking values in the tropical semiring:

$$\chi(X) = \omega^{\dim(X)}$$
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If $M$ is a model of a geometric theory, there is an Euler characteristic taking values in the tropical semiring:

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The existence of the dimension theory can be proven by checking definability and Fubini for the measure

$$\mu(X) = \begin{cases} 
0 & \text{if } X = \emptyset \\
\omega & \text{if } |X| \geq \aleph_0 \\
1 & \text{otherwise}
\end{cases}$$
A strongly minimal structure $M$ is \textit{unimodular} if whenever $a \in \text{acl}(Sb)$ and $b \in \text{acl}(Sa)$ but $a, b \notin \text{acl}(S)$,

$$\text{mult}(a/Sb) = \text{mult}(b/Sa)$$

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where the multiplicity of an algebraic type is the number of realizations.

Pseudo-finite and $\aleph_0$-categorical strongly minimal structures are known to be unimodular.
Remark

In a unimodular strongly minimal structure, there is a semiring-valued Euler characteristic given by

\[ \chi(X) = Z(X)\omega^{\dim(X)} \]

where \( \dim(X) \) is the Morley rank of \( X \) and \( Z(X) \) is the Zilber degree.
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The relevant semiring can be constructed by taking the free semiring \( \mathbb{N}[\omega] \) on one generator \( \omega \), modulo the equivalence relation of having the same leading term.
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The relevant semiring can be constructed by taking the free semiring \( \mathbb{N}[\omega] \) on one generator \( \omega \), modulo the equivalence relation of having the same leading term. For example,

\[
2\omega^3 \cdot 3\omega^2 = 6\omega^5 \\
2\omega^3 + 3\omega^2 = 2\omega^3 \\
2\omega^3 + 3\omega^3 = 5\omega^3
\]
Let $M$ be an ultraproduct of finite structures. Let $\mathbb{Z}^*$ be the corresponding ultrapower of $\mathbb{Z}$. 

Every definable set $X$ in $M$ has a nonstandard "size" $|X| \in \mathbb{Z}^*$. The map $X \mapsto |X|$ is a $\mathbb{Z}^*$-valued strong Euler characteristic.

If $N > 0$, then $\mathbb{Z}^*/N \cong \mathbb{Z}/N$. The map $\chi_N(X) := |X| + N \mathbb{Z}^* \in \mathbb{Z}^*/N \cong \mathbb{Z}/N$ defines a $\mathbb{Z}/N$-valued Euler characteristic $\chi_N$. 

Will Johnson  
Pseudo-finite fields  
October 18, 2018  
27 / 70
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1. It is a general fact that if $\chi$ is a definable and strong $R$-valued Euler characteristic, and $\phi : R \to S$ is a ring homomorphism, then $\phi \circ \chi$ is a definable and strong $R$-valued Euler characteristic.
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3. It is a general fact that an ultraproduct of strong Euler characteristics is strong.
**Remark**

*The Euler characteristic $\chi_N$ is strong.*

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4. Therefore, $\chi_N$ is strong on ultraproducts of finite structures.
If $N$ divides $M$, then

$$
\chi_N(X) \equiv \chi_M(X) \mod N
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Consequently, the $\chi_N$ for all $N$ can be assembled into a single $\hat{\chi}(X)$ taking values in the ring

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\hat{\mathbb{Z}} := \lim_{\leftarrow N} \mathbb{Z}/N.
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Let \( \pi_N \) be the natural map \( \hat{\mathbb{Z}} \to \mathbb{Z}/N \).

**Definition (abuse of notation)**

A \( \hat{\mathbb{Z}} \)-valued Euler characteristic \( \hat{\chi} \) is **strong** if \( \pi_N \circ \hat{\chi} \) is strong for every \( N \).

A \( \hat{\mathbb{Z}} \)-valued Euler characteristic \( \hat{\chi} \) is **definable** if \( \pi_N \circ \hat{\chi} \) is definable for every \( N \).
Remark

For definable and strong $\hat{\mathbb{Z}}$-valued $\hat{\chi}$, the integration formula still holds

$$\hat{\chi} \left( \bigsqcup_{x \in X} D_x \right) = \int_{x \in X} \hat{\chi}(D_x) \, d\hat{\chi}(x),$$

but we need to use $p$-adic integration.
If $M$ is an ultraproduct of finite structures, then there is canonically a $\hat{\mathbb{Z}}$-valued Euler characteristic $\hat{\chi}$ that is strong, but not necessarily definable.
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**Theorem (W. Johnson)**

*If $K$ is an ultraproduct of finite fields, then this $\hat{\chi}$ is definable.*
If \((M, \prec)\) is an ultraproduct of finite total orders, then \(\chi_2\) isn’t definable:

\[
\chi(x) = \begin{cases} 
0 & \text{if } x \text{ is the smallest element of } M, \\
1 & \text{otherwise.}
\end{cases}
\]

Let \(-\infty\) be the smallest element of \(M\), and let \(I_x = [-\infty, x]\). Then \(\chi_2(I_x)\) needs to alternate between 0 and 1, because:

\[
\chi_2(I_x) = \chi_2(I_{x-1}) + \chi_2(\{x\}) = \chi_2(I_{x-1}) + 1.
\]

No definable set has this property (use quantifier elimination or consider automorphisms of \(\aleph_0\)-saturated models).
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The theory of finite fields

Let $\mathcal{L}_{\text{rings}}$ be the first-order language of rings.

**Definition**

The (elementary) theory of finite fields is the set $T_{ff}$ of sentences $\varphi \in \mathcal{L}_{\text{rings}}$ such that $K \models \varphi$ for every finite field $K$.
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**Example**

- The map $x \mapsto x(x - 1)$ isn’t injective on any field. On finite fields, it can’t be surjective, so

  $$\exists y \forall x : y \neq x(x - 1)$$

  is in $T_{ff}$.

- The sentence $\forall x \exists y : x = y^2$ (every element is a square) is true in some finite fields, and false in others. This sentence is not in $T_{ff}$. 

Note: $T_{ff}$ is not complete.
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Fields of the second type are called *pseudo-finite fields*. The following transfer principles hold:

- If $\varphi \in \mathcal{L}_{rings}$ holds in infinitely many finite fields, then $\varphi$ holds in some pseudo-finite fields.
- If $\varphi \in \mathcal{L}_{rings}$ holds in almost all finite fields, then $\varphi$ holds in all pseudo-finite fields.
In *The Elementary Theory of Finite Fields*, Ax wrote down a computable set of axioms and proved that these axioms generate $T_{ff}$.
James Ax’s work

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From this, Ax proved:

**Corollary**

$T_{ff}$ is computable: there is an algorithm which takes $\varphi \in \mathcal{L}_{\text{rings}}$ and determines whether $\varphi$ holds in all finite fields.

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**Proof.**

$T_{ff}$ is computably enumerable (CE) because it is generated by a computable set of axioms. $T_{ff}$ is co-CE because one can inspect individual finite fields one by one to look for counterexamples.
Ingredients in Ax’s proof

Let $T_{Ax}$ denote Ax’s axioms for the theory of finite fields. The proof that $T_{Ax} \vdash T_{ff}$ uses three ingredients:

1. Weil’s Riemann Hypothesis for function fields: used to show that finite and pseudo-finite fields satisfy $T_{Ax}$.
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1. **Weil’s Riemann Hypothesis for function fields**: used to show that finite and pseudo-finite fields satisfy $\mathcal{T}_{Ax}$.

2. **A model theoretic argument**: used to characterize elementary equivalence between fields satisfying $\mathcal{T}_{Ax}$.

3. **The Chebotarev density theorem**: used to show that every model of $\mathcal{T}_{Ax}$ is elementarily equivalent to a pseudo-finite field.

1 and 3 are moderately deep results from algebraic geometry and number theory, respectively.
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Ax’s characterization of pseudo-finite fields

From the axioms that generate $T_{ff}$, Ax gave the following algebraic characterization of pseudo-finite fields:

**Corollary**

A field $K$ is pseudo-finite exactly if it satisfies the following algebraic conditions:

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3. $K$ is pseudo-algebraically closed (PAC)
Two definitions of PAC

For model theorists:

**Definition**

A field $K$ is *perfect and PAC* if, when we embed $K$ into a monster model $\mathcal{M}$ of ACF, every stationary type over $K$ is finitely satisfiable in $K$.

Remark

It suffices to check the stationary types of Morley rank 1.

For algebraic geometers:

**Definition**

A field $K$ is *PAC* if for every geometrically integral finite type $K$-scheme $V$, the set $V(K)$ of $K$-rational points is non-empty.

Remark

It suffices to check 1-dimensional $V$ (i.e., algebraic curves).
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Surprisingly pseudo-finite fields

- If $\sigma$ is chosen randomly in $\text{Gal}(\mathbb{Q}) := \text{Aut}(\mathbb{Q}^{\text{alg}} / \mathbb{Q})$ with respect to Haar measure, then the field of elements fixed by $\sigma$ is pseudo-finite with probability 1.
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If $(\mathbb{Z}, +, \cdot)$ is a model of Peano arithmetic, and $(p)$ is a principal prime ideal, then $\mathbb{Z}/(p)$ is a pseudo-finite field.
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- Most subfields of \( \mathbb{F}_p^{\text{alg}} \) are pseudo-finite. For example, the subfield generated by

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\mathbb{F}_p^2 \cup \mathbb{F}_p^3 \cup \mathbb{F}_p^5 \cup \cdots
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  \]
  is pseudo-finite.

- If $\mathbb{A}_\mathbb{Q}$ is the number theorist’s ring of adeles and $\mathfrak{m}$ is any maximal ideal other than the “obvious” ones, then quotient $\mathbb{A}_\mathbb{Q}/\mathfrak{m}$ is a pseudo-finite field.
Theorem (W. Johnson)

Every pseudo-finite field $K$ admits a definable and strong $\hat{\mathbb{Z}}$-valued Euler characteristic $\hat{\chi}$. 

This $\hat{\chi}$ isn't uniquely determined, unfortunately. It depends on a choice of an element $\sigma \in \text{Aut}(K_{\text{alg}}/K)$. 
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Proposition

If $K$ is an algebraically closed field of positive characteristic, then $K$ admits no $R$-valued strong Euler characteristics for any nontrivial ring $R$.

The proof involves examining a couple explicit maps from $K$ to $K$ and proving that $1 = 0$ in $R$. 

Strong Euler characteristics needn’t exist
Let $\mathcal{L}^+_{\text{rings}}$ be the first-order language of rings expanded with new quantifiers $\mu^n_k x$, where

$$\mu^n_k x : P(x) \iff$$

the number of $x$ such that $P(x)$ holds is congruent to $k \mod n$
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For example,

- $\mu^2_0 x$ means “there exists an even number of $x$ such that...”
- $\mu^2_1 x$ means “there exists an odd number of $x$ such that...”
Let \( \mathcal{L}_{\text{rings}}^+ \) be the first-order language of rings expanded with new quantifiers \( \mu_k^n x \), where

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\mu_k^n x : P(x) \iff \text{the number of } x \text{ such that } P(x) \text{ holds is congruent to } k \mod n
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For example,

- \( \mu_0^2 x \) means “there exists an even number of \( x \) such that…”
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These quantifiers make sense in any finite structure.
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These quantifiers make sense in any finite structure.

**Main Theorem (W. Johnson)**

*There is an algorithm which takes a sentence $\varphi \in \mathcal{L}_{\text{rings}}^+$ and determines whether $\varphi$ holds in every finite field.*
Eliminating the $\mu^n_k$, or not

Reduce to Ax’s decidability result by eliminating the $\mu^n_k$ quantifiers.
Eliminating the $\mu_k^n$, or not

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Example

$\mu_1^2 x \exists y : x = y^2$ (there are an odd number of perfect squares)
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Example

$\mu_1^2 x \exists y : x = y^2$ (there are an odd number of perfect squares) is equivalent to

$$(1 + 1 \neq 0) \land (\exists x : x^2 = -1)$$
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Reduce to Ax’s decidability result by eliminating the $\mu^k_n$ quantifiers.

Example

$\mu^2_1 x \exists y : x = y^2$ (there are an odd number of perfect squares) is equivalent to

$$(1 + 1 \neq 0) \land (\exists x : x^2 = -1)$$

Example

$\mu^5_1 x : x = x$ (size of the field is congruent to 1 mod 5)

Non-example

$\mu^3_1 x : x = x$ (size of the field is congruent to 3 mod 5) is not equivalent to any sentence in $L_{\text{rings}}$. We need an expanded structure.
Eliminating the $\mu^*_k$, or not

Reduce to Ax’s decidability result by eliminating the $\mu^*_k$ quantifiers.

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$\mu_1^2 x \exists y : x = y^2$ (there are an odd number of perfect squares) is equivalent to

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Example

$\mu_1^5 x : x = x$ (size of the field is congruent to 1 mod 5) is equivalent to

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Non-example

$\mu_3^5 x : x = x$ (size of the field is congruent to 3 mod 5)
Eliminating the $\mu^m_k$, or not

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Will Johnson  
October 18, 2018 43 / 70
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A difference field $(K, \sigma)$ is a field $K$ and an endomorphism $\sigma : K \rightarrow K$. 

**Definition**
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A difference field $(K, \sigma)$ is a field $K$ and an endomorphism $\sigma : K \to K$.

Why?
“Difference” fields

Differentiation satisfies these rules:

\[
\partial(f \cdot g) = f \partial g + g \partial f
\]
\[
\partial(f + g) = \partial f + \partial g
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“Difference” fields

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Definition

A differential field is a field \( K \) and an operator \( \partial : K \to K \) satisfying the product and sum rules for differentiation.
The difference operator $(\delta f)(x) = f(x + 1) - f(x)$ satisfies analogous rules:

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\delta(f \cdot g) = f \delta g + g \delta f + (\delta f)(\delta g)
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\begin{align*}
\delta(f \cdot g) &= f\delta g + g\delta f + (\delta f)(\delta g) \\
\delta(f + g) &= \delta f + \delta g
\end{align*}
\]

A difference field should be a field \(K\) and an operator \(\delta : K \rightarrow K\) satisfying the product and sum rules for differences.

**Remark**

*The axioms of \(\delta\) are an obfuscated way of saying that \(x \mapsto x + \delta x\) is a field endomorphism.*
In characteristic $p$, the following identities hold:

$$(x + y)^p = x^p + y^p$$

$$(xy)^p = x^p \cdot y^p$$

In particular, $x \mapsto x^p$ is a field endomorphism. More generally, $x \mapsto x^q$ is a field endomorphism for any power $q = p^k$. 

Definition

For $q = p^k$, the $q$th Frobenius difference field is $\mathbb{F}_{alg}$ with $\sigma$ given by $\sigma(x) = x^q$.

Definition

The (elementary) theory of Frobenius difference fields is the set $T_{frob}$ of $\phi$ holding in every Frobenius difference field.
The theory of Frobenius difference fields

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The (elementary) theory of Frobenius difference fields is the set $T_{\text{frob}}$ of $\varphi$ holding in every Frobenius difference field.
In *The Elementary Theory of the Frobenius Automorphism* (149 pages), Hrushovski wrote down a computable set of axioms and (probably) proved that these axioms generate $T_{\text{frob}}$.

**Corollary**

$T_{\text{frob}}$ is computable: there is an algorithm which takes $\phi$ in the language of difference fields and determines whether $\phi$ holds in all finite fields.

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The models of $T_{\text{frob}}$ are the Frobenius difference fields plus the models of ACFA.
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**Corollary**

The models of $T_{\text{frob}}$ are the Frobenius difference fields plus the models of ACFA.
Proposition

The theory of difference fields has a model companion, called ACFA.

This means:

The models of ACFA are exactly the existentially closed difference fields.

Every difference field can be embedded into a model of ACFA.

If $K \leq L$ is an inclusion of models of ACFA, then $K \preceq L$. 
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This means:

- The models of ACFA are exactly the existentially closed difference fields.
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- If $K \leq L$ is an inclusion of models of ACFA, then $K \preceq L$. 
ACFA and pseudo-finite fields

Definition

If \((K, \sigma)\) is a difference field, the **fixed field** \(K^\sigma\) is

\[
K^\sigma := \{ x \in K \mid \sigma(x) = x \}
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ACFA and pseudo-finite fields

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ACFA turns out to be closely related to pseudo-finite fields.

Proposition

- If \((K, \sigma) \models ACFA\), then \(K^\sigma\) is pseudo-finite.
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- If \((K, \sigma) \models \text{ACFA}\), then \(K^\sigma\) is pseudo-finite.
- Up to elementary equivalence, all pseudo-finite fields arise this way.
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- If \((K, \sigma)\) is a Frobenius difference field, then \(K^\sigma\) is a finite field.
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Consequently, Hrushovski’s work **generalizes** Ax’s work.
We need a (much easier) variant of Hrushovski’s work on ACFA and Frobenius difference fields.
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**Definition**

A difference field \((K, \sigma)\) is *periodic* if every \(x \in K\) has finite orbit under \(\sigma\), i.e., for every \(x \in K\) there is some \(n \geq 1\) such that \(\sigma^n x = x\).
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**Example**

- If \(\sigma \in \text{Aut}(\mathbb{Q}^{\text{alg}} / \mathbb{Q})\), then \((\mathbb{Q}^{\text{alg}}, \sigma)\) is periodic.
- Frobenius difference fields are periodic.
Problem: periodic difference fields are not an elementary class.

Solution: work in a multi-sorted structure ($K_1, K_2, K_3, \ldots$), where $K_i$ is the fixed field of $\sigma_i$. Each $K_i$ has the structure of a difference field. There are inclusion maps $\iota_n, m: K_n \rightarrow K_m$ whenever $n \mid m$.

Remark: In this language, periodic difference fields are now an elementary class. In this language, Frobenius difference fields are essentially finite: every definable set is finite.

To highlight the change of language, we will refer to periodic difference fields as periodic fields henceforth.
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Let $T_{frob}$ denote the (elementary) theory of Frobenius periodic fields.
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1. There is a computable set of axioms which generate $T_{frob}$.
2. Therefore, $T_{frob}$ is computable.
3. The theory of periodic difference fields has a model companion “ACPF.”
4. The models of $T_{frob}$ are the Frobenius periodic fields and the models of ACPF.

All of these facts were probably initially discovered by Hrushovski, prior to his work on ACFA.
ACPF and pseudo-finite fields

ACPF is closely linked to pseudo-finite fields:

**Proposition**

1. **If** $(K, \sigma)$ **is a model of ACPF**, then $K^\sigma$ is a pseudo-finite field.
2. *All pseudo-finite fields arise this way.*
ACPF is closely linked to pseudo-finite fields:

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1. If \((K, \sigma)\) is a model of ACPF, then \(K^\sigma\) is a pseudo-finite field.
2. All pseudo-finite fields arise this way.

More explicitly, if \(K\) is pseudo-finite then \(\text{Aut}(K^{\text{alg}}/K)\) is isomorphic to \(\hat{\mathbb{Z}}\). If \(\sigma\) is any topological generator of \(\hat{\mathbb{Z}}\), then \((K^{\text{alg}}, \sigma)\) is a model of ACPF with fixed field \(K\).
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**Proposition**

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More explicitly, if \(K\) is pseudo-finite then \(\text{Aut}(K^{\text{alg}}/K)\) is isomorphic to \(\hat{\mathbb{Z}}\). If \(\sigma\) is any *topological generator* of \(\hat{\mathbb{Z}}\), then \((K^{\text{alg}}, \sigma)\) is a model of ACPF with fixed field \(K\).

Consequently, the previous slide generalizes Ax’s work.
Today’s main theorem

Main Theorem (W. Johnson)

Let $(K, \sigma)$ be a model of ACPF. Then there is a canonical $\hat{\mathbb{Z}}$-definable Euler characteristic $\hat{\chi}$ on $(K, \sigma)$, which agrees with the pseudo-finite $\hat{\chi}$ if $(K, \sigma)$ is an ultraproduct of Frobenius periodic fields.
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Let $(K, \sigma)$ be a model of ACPF. Then there is a canonical $\hat{\chi}$ 0-definable Euler characteristic $\hat{\chi}$ on $(K, \sigma)$, which agrees with the pseudo-finite $\hat{\chi}$ if $(K, \sigma)$ is an ultraproduct of Frobenius periodic fields.

Theorem

The $\mu^k_n$ quantifiers can be uniformly eliminated across Frobenius periodic fields.
Earlier, we considered the statement that the number of elements in the field is 3 mod 5:

$$\mu_3^5 : x = x,$$

Interpreting the finite field as a fixed field of a Frobenius periodic field, the analogous statement for Frobenius periodic fields is

$$\mu_3^5 \in K_1 : x = x.$$
Earlier, we considered the statement that the number of elements in the field is 3 mod 5:

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Interpreting the finite field as a fixed field of a Frobenius periodic field, the analogous statement for Frobenius periodic fields is

$$\mu_3^5x \in K_1 : x = x.$$
Example: counting points in the fixed field

Let \( \{1, \omega, \omega^2, \omega^3, \omega^4\} \) be the 5th roots of unity. In the \( q \)th Frobenius periodic field, \( \sigma(x) = x^q \), and \( q = |K_1| \).
Let \( \{1, \omega, \omega^2, \omega^3, \omega^4\} \) be the 5th roots of unity. In the \( q \)th Frobenius periodic field, \( \sigma(x) = x^q \), and \( q = |K_1| \). Therefore:

- If \( |K_1| \equiv 1 \mod 5 \), then \( \sigma \) fixes \( \omega \)
- If \( |K_1| \equiv 2 \mod 5 \), then \( \sigma \) maps
  \[
  \omega \mapsto \omega^2 \mapsto \omega^4 \mapsto \omega^8 = \omega^3 \mapsto \omega^6 = \omega
  \]
- If \( |K_1| \equiv 3 \mod 5 \), then \( \sigma \) maps
  \[
  \omega \mapsto \omega^3 \mapsto \omega^9 = \omega^4 \mapsto \omega^{12} = \omega^2 \mapsto \omega^6 = \omega
  \]
- If \( |K_1| \equiv 4 \mod 5 \), then \( \sigma \) maps
  \[
  \omega \mapsto \omega^4 \mapsto \omega^{16} = \omega
  \]
Consequently,

\[
\left( \mu_3^5 x \in K_1 : x = x \right) \iff \\
\left( \exists x \in K_4 : x^5 = 1 \land x \neq 1 \land \sigma(x) = x^3 \right)
\]
Consequently,

\[(\mu_3^5 \in K_1 : x = x) \iff (\exists x \in K_4 : x^5 = 1 \land x \neq 1 \land \sigma(x) = x^3)\]

In general, \(\chi_N(K_1)\) is determined by examining how \(\sigma\) acts on primitive \(N\)th roots of unity
Consequently,

\[ (\mu_3^5 x \in K_1 : x = x) \iff (\exists x \in K_4 : x^5 = 1 \land x \neq 1 \land \sigma(x) = x^3) \]

In general, \( \chi_N(K_1) \) is determined by examining how \( \sigma \) acts on primitive \( N \)th roots of unity

\ldots unless \( N \) is divisible by the field characteristic.
In characteristic 2, what is $|K_1| \mod 256$?
In characteristic 2, what is $|K_1| \mod 256$?

- It’s 2 if $|K_1| = 2$.
- It’s 4 if $|K_1| = 4$.
- ... 
- It’s 128 if $|K_1| = 128$.
- It’s 0 otherwise.
In characteristic 2, what is $|K_1| \mod 256$?

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- It’s 128 if $|K_1| = 128$.
- It’s 0 otherwise.

So $\chi_{256}(K_1)$ is definable.
In characteristic 2, what is $|K_1| \mod 256$?
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- $\ldots$
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In characteristic 2, what is $|K_1| \mod N$?
In characteristic 2, what is $|K_1|$ mod 256?

- It’s 2 if $|K_1| = 2$.
- It’s 4 if $|K_1| = 4$.
- ... (omitted)
- It’s 128 if $|K_1| = 128$.
- It’s 0 otherwise.

So $\chi_{256}(K_1)$ is definable.

In characteristic 2, what is $|K_1|$ mod $N$?

1. Write $N = N_0 \cdot 2^k$ with $N_0$ odd.
2. Use the approach above to determine $\chi_{2^k}(K_1)$.
3. Use roots of unity to determine $\chi_{N_0}(K_1)$. 

$\mathbb{Z}/N\mathbb{Z} \cong \left(\mathbb{Z}/N_0\mathbb{Z}\right) \times \left(\mathbb{Z}/2^k\mathbb{Z}\right)$. 

Will Johnson

Pseudo-finite fields

October 18, 2018
Bad characteristic

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So $\chi_{256}(K_1)$ is definable.

In characteristic 2, what is $|K_1| \mod N$?

1. Write $N = N_0 \cdot 2^k$ with $N_0$ odd.
2. Use the approach above to determine $\chi_{2^k}(K_1)$.
3. Use roots of unity to determine $\chi_{N_0}(K_1)$.
4. $\mathbb{Z}/N \cong (\mathbb{Z}/N_0) \times (\mathbb{Z}/2^k)$. 

Will Johnson
Pseudo-finite fields
October 18, 2018 59 / 70
Let $C$ be the definable set

$$C = \{(x, y) \in (K_1)^2 \mid x^2 - y^2 = 1\}$$

Question: how can we determine $\chi_N(C)$?
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Question: how can we determine $\chi_N(C)$?

Answer: $C$ is a \textit{genus 0 curve}, so it is birationally equivalent to the affine line. In this case, the function

$$ (x, y) \mapsto x - y $$

puts $C$ into definable bijection with $K_1$, except for errors at finitely many points.
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Question: how can we determine $\chi_N(C)$?

Answer: $C$ is a genus 0 curve, so it is birationally equivalent to the affine line. In this case, the function

$$(x, y) \mapsto x - y$$

puts $C$ into definable bijection with $K_1$, except for errors at finitely many points. Therefore, $\chi_N(C)$ is determined by $\chi_N(K_1)$. 
Elliptic curves

If $E$ is an elliptic curve (curve of genus 1), such as

$$y^2 = x^3 - x,$$

then there is a natural group structure on the points of $E$. 
Elliptic curves

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then there is a natural group structure on the points of $E$.

**Proposition (Folk theorem)**

When $N$ is prime to the characteristic, the mod $N$ value of $|E(\mathbb{F}_q)|$ is determined by how the $q$th-power Frobenius automorphism acts on the $N$th roots of unity and the $N$-torsion in the group $E(\mathbb{F}_q^{alg})$. 
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This allows us to prove uniform definability of $\chi_N(E(K_1))$ by breaking into cases according to how $\sigma$ acts on the $N$-torsion in $E(K)$ and $K^\times$. 
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This allows us to prove uniform definability of $\chi_N(E(K_1))$ by breaking into cases according to how $\sigma$ acts on the $N$-torsion in $E(K)$ and $K^\times$.

**Remark**

In the case of bad characteristic $p|N$, this approach still works, but is much more complicated and involves non-reduced finite group schemes.
If $C$ is an algebraic curve of genus $> 1$, then there is a canonically associated definable group $J$, called the *Jacobian*. 
Higher genus curves

1. If $C$ is an algebraic curve of genus $> 1$, then there is a canonically associated definable group $J$, called the Jacobian.

2. $\chi_N(C)$ is determined by examining how $\sigma$ acts on the $N$-torsion in the multiplicative group and the Jacobian.

3. For $N$ prime to the characteristic, this again follows from folk theorems in arithmetic geometry.

4. For $N = p^k$, one needs to carefully analyze non-reduced finite group schemes.
Example: the set of squares

Let $D$ be the definable set of perfect squares in $K_1$, i.e.,

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Then

$$\hat{\chi}(D) = \begin{cases} 
\hat{\chi}(K_1), & \text{in characteristic 2} \\
\frac{\hat{\chi}(K_1) + 1}{2}, & \text{otherwise}
\end{cases}$$
Example: the set of squares

Let $D$ be the definable set of perfect squares in $K_1$, i.e.,

$$D = \{ x^2 \mid x \in K_1 \}$$

Then

$$\hat{\chi}(D) = \begin{cases} 
\hat{\chi}(K_1) & \text{in characteristic 2} \\
\frac{\hat{\chi}(K_1)+1}{2} & \text{otherwise}
\end{cases}$$

Note the utility of working with $\hat{\chi}$ rather than one of the individual $\chi_N$: division by 2 wouldn’t be well defined in $\mathbb{Z}/4$. 
General 1-dimensional definable sets

General definable subsets of $K_1$ are like the set of squares:

**Proposition**

*If $X \subseteq K_1$ is definable, then there is an algebraic curve $C$ defined over $K_1$, and an algebraic map $f : C \rightarrow K_1$ defined over $K_1$ such that $X = f(C(K_1))$.***
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This fact follows on general model-theoretic grounds by combining

- Model completeness of ACPF
- The ability to amalgamate periodic difference fields over bases that are algebraically closed.
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**Example**

If $D$ is the set of squares in $K_1$, then take $C = K_1$ and $f : K_1 \to K_1$ given by $f(x) = x^2$. 
Proposition

If $X \subseteq K_1$ is definable, then there is an algebraic curve $C$ defined over $K_1$, and an algebraic map $f : C \rightarrow K_1$ defined over $K_1$ such that $X = f(C(K_1))$.

Proposition

In the situation above, $\hat{\chi}(X)$ is determined in an explicit way by the values $\hat{\chi}(C), \hat{\chi}(C \times_{K_1} C), \hat{\chi}(C \times_{K_1} C \times_{K_1} C), \ldots$
The overall proof strategy is to define the $\chi_N$ uniformly across Frobenius periodic fields and models of ACPF by explicitly writing down axioms.

1. Each $\chi_N$ is a strong $\mathbb{Z}/N\mathbb{Z}$-valued Euler characteristic.
2. If $N | M$, then $\chi_N(X) \equiv \chi_M(X) \mod N$.
3. If $C$ is an algebraic curve, then $\chi_N(C)$ is given by an explicit black box from arithmetic geometry, involving $N$-torsion in the Jacobian and the multiplicative group.
Endgame: Beth implicit definability

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After writing down axioms, one then shows

1. The axioms are true in Frobenius periodic fields, hence in ultraproducts of Frobenius periodic fields.
2. In models of ACPF, the axioms determine at most one $\hat{\chi}|K_1$. 

By Beth implicit definability, it follows that $\hat{\mu} := \hat{\chi}|K_1$ is 0-definable (when it exists).

Because $\hat{\chi}$ is strong, it follows that $\hat{\chi}$ must be definable, if it exists.

If $K$ is a model of ACPF, then $K \equiv L$ for some ultraproduct $L$ of Frobenius periodic fields. 

The Euler characteristic $\hat{\chi}$ exists and is 0-definable in $L$, and therefore exists and is 0-definable in $K$. 

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4. Because \( \hat{\chi} \) is strong, it follows that \( \hat{\chi} \) must be definable, if it exists.

5. If \( K \) is a model of ACPF, then \( K \equiv L \) for some ultraproduct \( L \) of Frobenius periodic fields.

6. The Euler characteristic \( \hat{\chi} \) exists and is 0-definable in \( L \), and therefore exists and is 0-definable in \( K \).
Directions for future research

Can we prove definability of $\chi_N$ in these other pseudo-finite settings?

- Finite-rank sets in ACFA
- Non-standard integers modulo an infinite integer.
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Can we prove definability of $\chi_N$ in these other pseudo-finite settings?
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- Non-standard integers modulo an infinite integer.

Are there any connections to $p$-adic L-functions?


I would like to thank the organizers for inviting me.
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Questions?