More combinatorics and and topologies on ω_1

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Strong negative partition relations

Definition

$Pr_0(\kappa, X, (\theta : \lambda))$ *asserts that there is a function* $c : [\kappa]^2 \to X$ *such that whenever we are given* $\alpha < \theta$, $\beta < \lambda$, *two families* $\mathscr{A} \subset [\kappa]^\alpha$, $\mathscr{B} \subset [\kappa]^\beta$ *of* κ *many pairwise disjoint sets and a function* $h : \alpha \times \beta \to X$, *then there are* $a \in \mathscr{A}$, $b \in \mathscr{B}$ *such that* $a < b$ *and* $c(a(i), b(j)) = h(i, j)$ *for any* $i < \alpha, j < \beta$.

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Question

$Pr_0(\omega_1, \omega, (2:2)) \Rightarrow Pr_0(\omega_1, \omega, (2:3))$?

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More on *osc*

(P-Wu) For any $A \in [\omega_1]^{\omega_1}$ and an uncountable family of pairwise disjoint sets $\mathscr{B} \subset [\omega_1]^l$, there are $A' \in [A]^{\omega_1}$, $\mathscr{B}' \in [\mathscr{B}]^{\omega_1}$ and $\langle c_j : j < l \rangle \in \mathbb{Z}^l$ such that for any $\alpha \in A'$, for any $b \in \mathscr{B}'$, if $\alpha < \min b$, then $osc(\alpha, b(j)) = osc(\alpha, b(0)) + c_j$ for any $j < l$.

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(P-Wu) For any $A \in [\omega_1]^{\omega_1}$ and an uncountable family of pairwise disjoint sets $\mathscr{B} \subset [\omega_1]^2$, there are $A' \in [A]^{\omega_1}$, $\mathscr{B}' \in [\mathscr{B}]^{\omega_1}$ and $c \in \mathbb{Z}$ such that for any $\alpha \in A'$, for any $b \in \mathscr{B}'$, if $\alpha < \min b$, then
$osc(\alpha, b(1)) = osc(\alpha, b(0)) + c$.

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If the image is ω_1, is the projection (*mod* countable coordinates) still surjective for any uncountable set?

Definition

1. *A C-sequence is a sequence* $\langle C_\alpha : \alpha < \omega_1 \rangle$ *such that* $C_{\alpha+1} = \{\alpha\}$ *and* C_α *is a cofinal subset of* α *of order type* ω *for limit* α*'s.*
2. $\rho_1 : [\omega_1]^2 \to \omega$, *defined recursively by* $\rho_1(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}$ *with boundary value* $\rho_1(\alpha, \alpha) = 0$. $\rho_{1\beta} : \beta \to \omega$ *is defined by* $\rho_{1\beta}(\alpha) = \rho_1(\alpha, \beta)$ *for* $\alpha < \beta$.
3. *For any C-sequence, the lower trace* $L : [\omega_1]^2 \to [\omega_1]^{<\omega}$ *is recursively defined for any* $\alpha \leq \beta < \omega_1$ *as follows:*
    - $L(\alpha, \alpha) = 0$;
    - $L(\alpha, \beta) = (L(\alpha, \min(C_\beta \setminus \alpha)) \cup \{\max(C_\beta \cap \alpha)\}) \setminus \max(C_\beta \cap \alpha)$.

Definition

1. *For two functions* s, t *on a common finite set of ordinals* F, $Osc(s, t; F) = \{\alpha \in F \setminus \{\min F\} : s(\max F \cap \alpha) \leq t(\max F \cap \alpha)$ *and* $s(\alpha) > t(\alpha)\}$.
2. $osc : [\omega_1]^2 \to \omega$ *is defined by* $osc(\alpha, \beta) = |\{Osc(\rho_{1\alpha}, \rho_{1\beta}; L(\alpha, \beta))\}|$.

hereditarily eventually surjective projection

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For ω, a coloring has hereditarily eventually surjective projection iff it witnesses $Pr_0(\omega_1, \omega, (2:2))$.

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Theorem (PFA)

No.

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For a coloring $c : [\omega_1]^2 \to X$ ($|X| = \omega_1$), $Pj^c_\alpha(B) = \{c(\alpha, \beta) : \beta \in B \setminus \alpha\}$.

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A simple lifting gives the following.

Fact

There is a coloring $c : [\omega_1]^2 \to \omega_1$ such that for any σ-complete ideal $\mathcal{I}$, for any $B \in [\omega_1]^{\omega_1}$, there is an $\alpha < \omega_1$ such that any finite intersection of $\{Pj^c_\gamma(B) : \gamma > \alpha\}$ is in $\mathcal{I}^+$.

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Question

Given an L space/group X of size ω_1. Is there a coloring $c : [\omega_1]^2 \to X$ that is hereditarily eventually dense?

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Fact (MA_{ω_1})

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For any $X \in [\omega_1]^{\omega_1}$, for any $k, l < \omega$, for any $\langle c_{ij} : i < k, j < l\rangle \in \mathbb{Z}^{k \times l}$ such that $c_{i0} = 0$ for $i < k$, there are uncountable families $\mathscr{A} \subset [X]^k$, $\mathscr{B} \subset [X]^l$ that are pairwise disjoint and for any $a \in \mathscr{A}, b \in \mathscr{B}$, if $a < b$, then $osc(a(i), b(j)) = osc(a(i), b(0)) + c_{ij}$ for $i < k, j < l$.

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Corollary

If $c(\alpha, \beta) = f(frac(\theta_\alpha osc(\alpha, \beta)))$ *witnesses* $Pr_0(\omega_1, \omega, (\omega : n))$ *where* $f^{-1}[\omega \setminus \{0\}]$ *has non-empty interior, then* c *fails to witness* $Pr_0(\omega_1, \omega, (\omega : m))$ *for some* $m < \omega$.

A stronger property

(*) For any $X \in [\omega_1]^{\omega_1}$, for any $\langle c_{ij} : i, j < 2 \rangle \in \omega^{2\times 2}$ such that $c_{00} = c_{11} = 0$, there is an uncountable families $\mathscr{A} \subset X$ of pairwise disjoint sets such that for any $a < b \in \mathscr{A}$, $osc(a(i), b(j)) = osc(a(i), b(i)) + c_{ij}$ for $i, j < 2$.

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Theorem (P-Wu)

(*) is independent of ZFC.

topology not determined by countable sets

Theorem

For any $n < \omega$, there is an L space X witnessing $Pr_0(\omega_1, \omega, (\omega : n))$ such that every countable subset is closed and discrete.

Theorem

Suppose that G is a separable topological Ablien group satisfying one of the following properties.

- *There is a countable dense subgroup H,* $\forall g \in H \forall n \in \omega \setminus \{0\}\ \exists x \in H\ nx = g$.
- *G is metrizable and for any $n \in \omega \setminus \{0\}$, for any open set O, there exists $x \in G$ such that $nx \in O$.*

Then for any $n < \omega$, there is a topological group $X \subset G^{\omega_1}$ such that X^n is an L group.

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Then for any $n < \omega$, there is a topological group $X \subset G^{\omega_1}$ such that X^n is an L group.

Taking G in above theorem to be, e.g., l_2, we get the following.

Corollary

For any $n < \omega$, there is a topological group X such that X^n is an L group and every countable subset of X is closed discrete.

Thank you!