Automorphisms and mild extensions of NFU

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Apology and introduction

I had intended to speak about my recent work with Ali Enayat (Gothenburg) and Matt Kaufmann (Texas), published in "Largest initial segments pointwise fixed by automorphisms of models of set theory", that classifies the countable structures in the language of set theory that can appear as a largest initial segment of a model of set theory that is pointwise fixed by a non-trivial automorphism. Instead, I have decided to present a new, but simpler, application of the ultrapower technology utilised in the aforementioned paper, and also give a bit of background about how one can build models of extensions of the urelemented variant of Quine's "New Foundations" Set Theory NFU from models of set theory that admit non-trivial automorphism.

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Definition

Mac Lane Set Theory, Mac, *is the $\mathcal{L}$-theory with axioms: extensionality, pair, union, emptyset, powerset, infinity, transitive containment, Δ_0-separation, set foundation and the axiom of choice*

Notation and definitions

- The class $\Delta_0^{\mathcal{P}}$ is the smallest class of $\mathcal{L}$-formulae that contains all atomic formulae, contains all compound formulae formed using the connectives of first-order logic, and is closed under quantification in the form $\mathcal{Q}x \in y$ and $\mathcal{Q}x \subseteq y$ where x and y are distinct variables and $\mathcal{Q}$ is $\exists$ or $\forall$

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- If $\mathcal{L}' \supseteq \mathcal{L}$ then we use $\Delta_0(\mathcal{L}')$ ($\Delta_0^{\mathcal{P}}(\mathcal{L}')$) to denote the smallest class of formulae that contains all atomic formulae, all compound formulae formed using the connectives of first-order logic, and is closed under quantification in the form $\mathcal{Q}x \in t$ (and $\mathcal{Q}x \subseteq t$) where t is an $\mathcal{L}'$-term and x is a variable that does not appear in t, and $\mathcal{Q}$ is $\exists$ or $\forall$

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- The classes $\Sigma_1^{\mathcal{P}}, \Pi_1^{\mathcal{P}}, \ldots, \Sigma_1(\mathcal{L}'), \Pi_1(\mathcal{L}'), \ldots$ and $\Sigma_1^{\mathcal{P}}(\mathcal{L}'), \Pi_1^{\mathcal{P}}(\mathcal{L}'), \ldots$ are defined inductively from the class $\Delta_0^{\mathcal{P}}$, $\Delta_0(\mathcal{L}')$ and $\Delta_0^{\mathcal{P}}(\mathcal{L}')$ in the same way as the classes $\Sigma_1, \Pi_1, \ldots$ are defined from Δ_0

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The terms of $\mathcal{L}_{\mathrm{NFU}}$ *are built-up inductively from variables using the function* $\langle \cdot, \cdot \rangle$*. Let* ϕ *be an* $\mathcal{L}_{\mathrm{NFU}}$*-formula. We use* **Term**(ϕ) *to denote the set of* $\mathcal{L}_{\mathrm{NFU}}$*-terms appearing in* ϕ*. We say that* $\sigma : \mathbf{Term}(\phi) \longrightarrow \mathbb{N}$ *is a stratification of* ϕ *if for all terms* s *and* t *appearing in* ϕ*,*

(i) *if* s *is a term appearing in* t *then* $\sigma(\text{'}t\text{'}) = \sigma(\text{'}s\text{'})$*,*

(ii) *if* $s \in t$ *is a subformula of* ϕ *then* $\sigma(\text{'}t\text{'}) = \sigma(\text{'}s\text{'}) + 1$*,*

(iii) *if* $s = t$ *is a subformula of* ϕ *then* $\sigma(\text{'}t\text{'}) = \sigma(\text{'}s\text{'})$*.*

If there exists a stratification of ϕ *then we say that* ϕ *is stratified.*

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NFU *(with infinity and choice) is the* $\mathcal{L}_{\mathrm{NFU}}$*-theory with axioms:*

(Weak Extensionality)
$\forall x \forall y(\mathcal{S}(x) \wedge \mathcal{S}(y) \Rightarrow (x = y \iff \forall z(z \in x \iff z \in y)))$

(Stratified Comprehension) for all stratified $\mathcal{L}_{\mathrm{NFU}}$*-formulae* $\phi(x, \vec{z})$,

$$\forall \vec{z} \exists y(\mathcal{S}(y) \wedge \forall x(x \in y \iff \phi(x, \vec{z})))$$

(Pairing) $\forall x \forall y \forall z \forall w(\langle x, y \rangle = \langle w, z \rangle \Rightarrow (x = w \wedge y = z))$

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- The Pairing Axiom implies that V is Dedekind infinite

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- In contrast to the relationship between ZF and ZFA (they are equivalent in a very strong sense), NFU is weaker than NF. In particular, Jensen (1969) shows that NFU is consistent and consistent with both the Axiom of Choice, and the negation of the Axiom of Infinity. Specker (1953) shows that NF proves the negation of the Axiom of Choice and the Axiom of Infinity.

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- Cardinals in NFU are represented as equivalence classes of equipollent sets (à la Frege and the representation of cardinals in ZF). Note that the existence of a universal set means that we do not need to use Scott's trick.
- Ordinals in NFU are represented as equivalence classes of isomorphic well-orderings
- If X is a set, then we use $|X|$ to denote the cardinal to which X belongs. If R is a well-ordering, then we use $[R]$ to denote the ordinal to which R belongs.

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- One strange feature of NFU (and NF) is the fact that not every set is the same size as its own set of singletons. We use ι to denote the singleton map $(x \mapsto \{x\})$ and write $\iota``X$ for the set of singletons of X $(\{\{x\} \mid x \in X\})$

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- For example, $|\iota``V| \neq |V|$, because if $|\iota``V| = |V|$, then the proof of Cantor's Theorem would tell us that $|V| < |\mathcal{P}(V)|$, which is clearly absurd $(\mathcal{P}(V) \subseteq V)$

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- Cantor's Theorem in its usual form is not provable in NFU. Instead we get that for all X,

$$|\iota``V| < |\mathcal{P}(V)|$$

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Definition

We say that a set X is **Cantorian** *if* $|X| = |\iota``X|$. *We say a set $X$ is* **strongly Cantorian** *if the restriction of the map $\iota$ to $X$ witnesses the fact that* $|X| = |\iota``X|$

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If X is a set then define

$$T(|X|) = |\iota``X|.$$

If R is a well-founded relation then define

$$T([R]) = [\{\langle \iota x, \iota y \rangle \mid \langle x, y \rangle \in R\}].$$

Mild extensions of NFU

The T operation restricts to an automorphism of the structure $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$, but this automorphism is not necessarily the identity. In other words, the set $\mathbb{N}$ is provably Cantorian, but not necessarily strongly Cantorian.

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$$\text{(Axiom of Counting } (\mathrm{AxCount})\text{): } (\forall n \in \mathbb{N})(T(n) = n)$$

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Building models of NFU

Let $\mathcal{M} \models \mathrm{Mac}$ and let $j : \mathcal{M} \longrightarrow \mathcal{M}$ be an automorphism such that there exists $c \in M$ with

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Define an $\mathcal{L}_{\mathrm{NFU}}$-structure $\mathcal{N} = \langle N, \in^{\mathcal{N}}, \mathcal{S}^{\mathcal{N}}, \langle \cdot, \cdot \rangle^{\mathcal{N}} \rangle$ by

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Theorem

(Jensen) $\mathcal{N} \models \mathrm{NFU}$

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Broadly speaking, the behaviour of the automorphism $j : \mathcal{M} \longrightarrow \mathcal{M}$ corresponds to the behaviour of the T operation in the model $\mathcal{N}$ of NFU defined on the previous slide. In particular:

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Theorem

(M. 2015)

(I) $\mathrm{NFU} + \mathrm{AxCount} \vdash \mathrm{Con}(\mathrm{NFU} + \mathrm{AxCount}_{\leq})$

(II) $\mathrm{NFU} + \mathrm{AxCount}_{\leq} \vdash \mathrm{Con}(\mathrm{NFU} + \mathrm{AxCount}_{\geq})$

What we know and don't know about these systems

Theorem

(M. 2017) $\mathrm{NFU} + \mathrm{AxCount}_{\geq} \vdash \mathrm{Con}(\mathrm{Mac})$, *and so* $\mathrm{NFU} + \mathrm{AxCount}_{\geq} \vdash \mathrm{Con}(\mathrm{NFU})$

What we know and don't know about these systems

Theorem

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Question

What are the exact strengths of the theories $\mathrm{NFU} + \mathrm{AxCount}_{\geq}$ *and* $\mathrm{NFU} + \mathrm{AxCount}_{\leq}$ *relative to subsystems of* ZFC*?*

Building models of set theory with automorphisms

Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be an $\mathcal{L}$-structure such that $|M| = \aleph_0$ and

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$\mathcal{M}_{\mathcal{F}} \models \Sigma_1(\mathcal{L}_{\mathcal{F}})\text{-}\mathit{collection} + \Delta_0(\mathcal{L}_{\mathcal{F}})\text{-}\mathit{separation}$

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There exists an ultrafilter $\mathcal{U} \subseteq (\mathcal{P}(\omega)^{\mathcal{M}})^*$ such that for all $n \in \omega$ and for all $f \in \mathcal{F}$, if

$$\mathcal{M} \models (\exists k \in \omega)(f : [\omega]^n \longrightarrow k)$$

then there exists $X \in \mathcal{U}$ such the $\hat{f}$ is constant on $[X^*]^n$.

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Let $\mathcal{U}$ be an ultrafilter that is obtained from the previous lemma.

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$$\phi(x_0, \ldots, x_n) \in T_{\mathcal{U}} \quad \text{if and only if} \quad \begin{array}{l} \exists X \in \mathcal{U} \text{ s.t.} \\ \mathcal{M}_{\mathcal{F}} \models \phi(a_0, \ldots, a_n) \\ \text{for all } a_0 < \cdots < a_n \in X^* \end{array}$$

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The structure of the ultrapower

Lemma

(a Łoś Theorem) If $\phi(x_0, \ldots, x_n)$ *a* Σ_1*-formula then for all* $[\tau_0], \ldots, [\tau_n] \in N$,

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- Let $m \in M$ and let $h_m \in \mathcal{F}$ be such that

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  These constant functions witness the fact that $\mathcal{M}$ embeds into $\mathcal{N}$

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Let $\phi(x_0, \ldots, x_n)$ be Π_2. For all $a_0, \ldots, a_n \in M$, if $\mathcal{M} \models \phi(a_0, \ldots, a_n)$, then

$$\mathcal{N} \models \phi([\hat{h}_{a_0}(c_0)], \ldots, [\hat{h}_{a_n}(c_0)])$$

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Observing that all the axioms of Mac except the powerset axiom can be written as Π_2-sentences yields:

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Let $f \in \mathcal{F}$ and let $i_0 < \cdots < i_n \in \mathbb{Z}$. If for all $j \in \mathbb{Z}$,

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then there exists $l \in M$ such that

$$\mathcal{N} \models ([\hat{h}_l(c_0)] \in [\hat{h}_\omega(c_0)]) \wedge ([\hat{f}(c_{i_0}, \ldots, c_{i_n})] = [\hat{h}_l(c_0)])$$

The automorphism of $\mathcal{N}$

The structure $\mathcal{N}$ admits a non-trivial automorphism. Define an automorphism $j : \mathcal{N} \longrightarrow \mathcal{N}$ by

$$j([\hat{f}(c_{i_0}, \ldots, c_{i_n})]) = [\hat{f}(c_{i_0+1}, \ldots, c_{i_n+1})]$$

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- The cofinality and "downward cofinality" of the class $\{[\mathrm{id}(c_i)] \mid i \in \mathbb{Z}\}$ ensures that for all $n \in N$, if $\mathcal{N} \models (n \in \omega)$, then $\mathcal{N} \models (j(n) \geq n)$

Conclusions and problems

Theorem

If $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ *is such that* $|M| = \aleph_0$ *and*

$$\mathcal{M} \models \mathrm{Mac} + \Delta_0\text{-}collection$$

Then $\mathcal{M}$ *has a cofinal extension* $\mathcal{N}$ *with* $\mathcal{N} \models \mathrm{Mac}$ *that admits a non-trivial automorphism* $j : \mathcal{N} \longrightarrow \mathcal{N}$ *such that for all* $n \in N$*, if* $\mathcal{N} \models (n \in \omega)$*, then* $\mathcal{N} \models (j(n) \geq n)$*.*

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(Mathias 2001, M. 2018) The theories Mac*,*
$\mathrm{Mac} + \Delta_0\text{-}\mathit{collection} + \Sigma_1\text{-}\mathit{separation}$ *and*
$\mathrm{Mac} + \Pi_1\text{-}\mathit{collection} + \Sigma_1\text{-}\mathit{separation}$ *are all equiconsistent.*

What does $\mathcal{M}$ need to satisfy in order to get a model of $\mathrm{NFU} + \mathrm{AxCount}_{\geq}$?

It is clear from the ultrapower construction of $\mathcal{N}$ that would get a point $c \in N$ such that

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if there exists $f \in M$ such that f is the map $n \mapsto \mathcal{P}^n(\omega)$. This yields:

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Theorem

(M. 2015) There is a model of $\mathrm{NFU} + \mathrm{AxCount}_{\geq}$ in which is the set of infinite cardinals is finite.

What does $\mathcal{M}$ need to satisfy in order to get a model of $\mathrm{NFU} + \mathrm{AxCount}_{\geq}$?

An assertion that appears to be closer to the strength of $\mathrm{NFU} + \mathrm{AxCount}_{\geq}$ is the assertion, in $\mathcal{M}$, that the restriction of the map $n \mapsto \mathcal{P}^n(\omega)$ to any natural number is a set. This assertion follows from the scheme of $\Pi_1^{\mathcal{P}}$-foundation.

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Question

Is there a way of modifying the ultrapower construction so that the resulting model is not cofinal in the original model?

Thank you!