Can Nonstandard Analysis Produce New Standard Theorems?

Renling Jin

(College of Charleston, South Carolina)

Fudan University, Shanghai
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OUTLINE:

1. Nonstandard model
2. Sumset phenomenon
3. Freiman's inverse problems
4. Buy-one-get-one-free scheme

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Definition of nonstandard model

$\mathbb{N}$: the set of all positive integers. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
$\mathbb{R}$: the set of all real numbers.
$\mathcal{R} = (\mathbb{R}; F)_{F \in \mathcal{F}}$: standard model of analysis where $\mathcal{F}$ is a collection of all finite dimensional real valued functions on $\mathbb{R}$.

Proposition (A. Robinson, 1963)

There is a structure $^*\mathcal{R} = (^*\mathbb{R}; {}^*F)_{F \in \mathcal{F}}$ *such that*

1. $\mathbb{R} \subseteq {}^*\mathbb{R}$ *and* $^*\mathbb{R} \setminus \mathbb{R} \neq \emptyset$,
2. *if* $\varphi(\overline{x}, \overline{F})$ *is a first-order formula in the language of* $\mathcal{R}$, *where* $\overline{x}$ *is a list of m variables and* $\overline{F}$ *is a list of n functions in* $\mathcal{F}$ *occurring in the formula, if* $\overline{a}$ *is a list of m real numbers, then* $\varphi(\overline{a}, \overline{F})$ *is true in* $\mathcal{R}$ *if and only if* $\varphi(\overline{a}, \overline{^*F})$ *is true in* $^*\mathcal{R}$.

The second part above is called the transfer principle.
We can require that $^*\mathcal{R}$ be countably saturated.

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First-order formula versus non-first-order formula

The following is a first-order formula:

$$\forall x, y\,(x \leqslant y \vee y \leqslant x)$$

The following is a formula which is not first-order:

$$\forall S \subseteq [0,1]\,(S \text{ has a least upper bound})$$

Let ${}^*\mathbb{N}$ be the set of all positive integers in ${}^*\mathcal{R}$. Then ${}^*\mathbb{N} \smallsetminus \mathbb{N} \neq \emptyset$. Elements in ${}^*\mathbb{N} \smallsetminus \mathbb{N}$ are called hyperfinite integers.

A hyperfinite integer is infinitely large but is finite from a nonstandard point of view.

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Standard part map and Loeb probability space

For any real number $r \in {}^*[0,1]$, there is a standard real number $\alpha \in [0,1]$ such that r is infinitesimally close to α. We define the standard part map *st* by letting $st(r) = \alpha$.

Let N be a hyperfinite integer and $\Omega = {}^*\mathbb{N} \cap [1, N]$. Let Σ_0 be the family of all *finite subsets of Ω and $\mu_0(A) = |A|/N$ for every $A \in \Sigma_0$. Then μ_0 is just the normalized counting measure on a *finite set Ω (in nonstandard sense).

For every $A \in \Sigma_0$ let

$$\mu(A) = st(\mu_0(A)).$$

Then (Ω, Σ_0, μ) is a standard finitely additive probability measure, which can be extended to a countably additive complete probability space (Ω, Σ, μ) called Loeb space.

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A taste

Let $[X]^2 := \{\{x, y\} : x, y \in X \text{ and } x \neq y\}$.

Ramsey Theorem for pairs

Given any $f : [\mathbb{N}]^2 \to \{0, 1\}$, there is an infinite $H \subseteq \mathbb{N}$ such that f is a constant function on $[H]^2$.

Proof: Fix a $C \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let $H_i \subseteq \mathbb{N}$ be maximal such that ${}^*f \upharpoonright [H_i \cup \{C\}]^2 \equiv i$ for $i = 0, 1$. We can assume that both H_0 and H_1 are finite because otherwise, we are done.

By transfer principle, there is a $c \in \mathbb{N}$ such that $c > \max H_i$ and $f \upharpoonright [H_i \cup \{c\}]^2 \equiv i$ for $i = 0, 1$. WLOG, let ${}^*f(\{c, C\}) = 0$. Then ${}^*f \upharpoonright [H_0 \cup \{c, C\}]^2 \equiv 0$, which contradicts the maximality of H_0.

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Existing results

Theorem (H. Steinhaus, 193?)

If A and B are two sets of real numbers with positive Lebesgue measure, then $A + B$ contains a nonempty open interval.

An infinite set $A \subseteq \mathbb{N}$ is syndetic if there is a finite set F of integers such that $\mathbb{N} \subseteq A + F$.

The upper (lower) Banach density $\overline{BD}$ ($\underline{BD}$) of A is defined by

- $\overline{BD}(A) = \lim_{n\to\infty} \sup_{k\geqslant 0} |A \cap [k+1, k+n]|/n$,
- $\underline{BD}(A) = \lim_{n\to\infty} \inf_{k\geqslant 0} |A \cap [k+1, k+n]|/n$

Theorem (H. Furstenberg, 1981)

If A is a set of positive integers with a positive upper Banach density, then $A - A = \{a - a' \mid a, a' \in A, a' \leqslant a\}$ is syndetic.

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Sumset phenomenon

Sumset phenomenon: If A and B are large in terms of "measure", then $A+B$ is not small in terms of "order-topology".

Let N be a hyperfinite integer and $\Omega = {}^*\mathbb{N} \cap [1, N]$. A set $U \subseteq \Omega \cup \{0\}$ is a cut if $0, 1 \in U$ and $U + U \subseteq U$.

A *finite set $A \subseteq \Omega$ is U-nowhere dense if for any $a, b \in \Omega$ with $b - a > U$, there exist $c, d \in \Omega$ such that $a < c < d < b$, $d - c > U$, and $[c, d] \cap A = \emptyset$.

A set $X \subseteq \Omega$ is U-meager if X is a countable union of U-nowhere dense sets.

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A theorem in the nonstandard model

Theorem (R. J., 2002)

Let N be a hyperfinite integer and $U \subseteq {}^*\mathbb{N} \cap [0, N]$ be a cut. If A and B are *finite subsets of $\Omega = {}^*\mathbb{N} \cap [1, N]$ with positive Loeb measure, then $A + B$ is not U-meager.

Reference: R. Jin, *Sumset phenomenon*, Proceedings of American Mathematical Society, Vol. 130, No. 3 (2002), 855–861.

Let $U_{\max} = \bigcap_{n \in \mathbb{N}} [0, N/n)$. Then $U_{\max}$ is a cut in Ω. In fact, $U_{\max}$ is the largest cut in Ω

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Let N be a hyperfinite integer and $U \subseteq {}^*\mathbb{N} \cap [0, N]$ be a cut. If A and B are *finite subsets of $\Omega = {}^*\mathbb{N} \cap [1, N]$ with positive Loeb measure, then $A + B$ is not U-meager.

Reference: R. Jin, *Sumset phenomenon*, Proceedings of American Mathematical Society, Vol. 130, No. 3 (2002), 855–861.

Let $U_{\max} = \bigcap_{n \in \mathbb{N}} [0, N/n)$. Then $U_{\max}$ is a cut in Ω. In fact, $U_{\max}$ is the largest cut in Ω

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Applications

Set $U = U_{\max}$. Then we have the following corollary:

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If A and B are sets of reals with positive Lebesgue density, then $A+B$ contains a non-empty open interval.

A set $A \subseteq \mathbb{N}$ is thick if it contains arbitrarily long intervals. A set A is piecewise syndetic if there is a finite set F such that $A+F$ is thick.

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Theorem (M. Beiglböck, V. Bergelson, A. Fish, H. Furstenberg, B. Weiss, 2006 – 2010)

Let G be a countable abelian amenable group and $A, B \subseteq G$.

- *If $\overline{BD}(A) > 0$ and $\overline{BD}(B) > 0$, then $A + B$ is piecewise Bohr.*
- *If C is piecewise Bohr, then there exist A and B such that $\overline{BD}(A) > 0$, $\overline{BD}(B) > 0$, and $A + B \subseteq C$.*

Definition

Suppose $A \subseteq \mathbb{N}$. Let

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{A(1,n)}{n} \text{ and } \overline{d}(A) := \limsup_{n\to\infty} \frac{A(1,n)}{n}.$$

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High density syndeticity

Most of the theorems in this section have generalizations in $\mathbb{Z}^n$.

Definition

For any $A \subseteq \mathbb{Z}$ and finite $E \subseteq \mathbb{Z}$ let $S_E(A) := \{x \in \mathbb{Z} : x + E \subseteq A\}$.

- *A is lower (upper) thick of level α if $\underline{d}(S_E(A)) \geq \alpha$ ($\overline{d}(S_E(A)) \geq \alpha$) for any finite set $E \subseteq \mathbb{Z}$.*
- *A is lower (upper) syndetic of level α if there is a finite set $F \subseteq \mathbb{Z}$ such that $F + A$ is lower (upper) thick of level α.*

Definition

A set A is strongly upper syndetic of level α if for any infinite set $S \subseteq \mathbb{Z}$, there exists a finite set F such that for any finite set $E \subseteq \mathbb{Z}$

$$\overline{d}_S(S_E(A+F)) := \limsup_{n \in S, n \to \infty} \frac{1}{n} |\{x \in [1, n] : x + E \subseteq A + F\}| \geq \alpha.$$

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Theorems

Let $A, B \subseteq \mathbb{Z}$.

Theorem (DGJLLM)

If $\overline{d}(A) = \alpha$ and $\overline{BD}(B) > 0$, then $A + B$ is upper syndetic of level α.

This theorem has been generalized to the setting of countable amenable groups.

Theorem (DGJLLM)

If $\underline{d}(A) = \alpha$ and $\overline{BD}(B) > 0$, then $A + B$ is lower syndetic of level $\alpha - \epsilon$ for any fixed $\epsilon > 0$.

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Examples

Example (DGJLLM)

There exist $A, B \subseteq \mathbb{Z}$ *such that* $\overline{d}(A) = 1/2$, $\overline{d}(B) = 1/2$, *and* $A + B$ *is upper syndetic of level* $1/2$ *but is not upper syndetic of level* $1/2 + \epsilon$ *for any* $\epsilon > 0$.

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There exist $A, B \subseteq \mathbb{Z}$ *such that* $\underline{d}(A) = 1/2$, $\overline{BD}(B) \geq 8/9$, *and* $A + B$ *is not lower syndetic of level* $1/2$.

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Let $A, B \subseteq \mathbb{N}$. If $\underline{d}(A) = \alpha$ and $\underline{d}(B) = \beta$, then $A + B$ is strongly upper syndetic of level $\min\{\alpha + \beta, 1\}$.

This theorem has no generalizations in $\mathbb{Z}^n$ yet.

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Freiman's theorems

Let A be a finite set of integers. Denote $2A := \{a + a' \mid a, a' \in A\}$.

Freiman's inverse phenomenon: If $2A$ is "small", then A should have some arithmetic structural properties.

Example: If $|2A| = 2|A| - 1$, then A is an arithmetic progression.

Theorem (G. A. Freiman, 1958)

If $|A| > 2$ and $|2A| = 2|A| - 1 + b$ for $0 \leqslant b < |A| - 2$, then A is a subset of an arithmetic progression of length at most $|A| + b$.

Theorem (G. A. Freiman, 1959)

If $|A| > 6$ and $|2A| = 3|A| - 3$, then A is either a subset of an arithmetic progression of length at most $2|A| - 1$ or A is the union of two arithmetic progressions with the same difference.

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Freiman's Conjecture

A set $B = I \cup J$ is called a bi-arithmetic progression if I and J are arithmetic progressions of the same difference and $2I, I+J, 2J$ are pairwise disjoint where $I + J = \{i + j \mid i \in I, j \in J\}$.

Conjecture (G. A. Freiman, 1961)

If A is sufficiently large and $|2A| = 3|A| - 3 + b$ for $0 \leq b < \frac{1}{3}|A| - 2$, then A is either a subset of an arithmetic progression of length at most $2|A| - 1 + b$ or a subset of a bi-arithmetic progression of length at most $|A| + b$.

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Inverse problem for upper density

Theorem (R. Jin, 2006)

Let $A \subseteq \mathbb{N}$ *with* $0 \in A$ *and* $\gcd(A) = 1$. *Suppose* $0 < \overline{d}(A) = \alpha < \frac{1}{2}$ *and* $\overline{d}(2A) = \frac{3}{2}\alpha$. *Then either*

- *there are* $g, m \in \mathbb{N}$ *with* $\gcd(g, m) = 1$ *such that* $A \subseteq \{0, m\} + g\mathbb{N}$ *and* $\alpha = \frac{2}{g}$ *or*
- *for any increasing sequence* $h_n \in \mathbb{N}$ *with* $\lim_{n\to\infty} \frac{A(h_n)}{h_n} = \alpha$, *there are* $0 < a_n < b_n < h_n$ *such that*
  1. $A \cap [a_n, b_n] = \emptyset$ *for each* n,
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Let $A \subseteq \mathbb{N}$ *with* $0 \in A$ *and* $\gcd(A) = 1$. *Suppose* $0 < \overline{d}(A) = \alpha < \frac{1}{2}$ *and* $\overline{d}(2A) = \frac{3}{2}\alpha$. *Then either*

- *there are* $g, m \in \mathbb{N}$ *with* $\gcd(g, m) = 1$ *such that* $A \subseteq \{0, m\} + g\mathbb{N}$ *and* $\alpha = \frac{2}{g}$ *or*
- *for any increasing sequence* $h_n \in \mathbb{N}$ *with* $\lim_{n\to\infty} \frac{A(h_n)}{h_n} = \alpha$, *there are* $0 < a_n < b_n < h_n$ *such that*
  1. $A \cap [a_n, b_n] = \emptyset$ *for each* n,
  2. $\lim_{n\to\infty} \frac{a_n}{h_n} = 0$, *and*
  3. $\lim_{n\to\infty} \frac{A(b_n, h_n)}{h_n - b_n + 1} = 1$.

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Reference: R. Jin, *Solution to the Inverse Problem for Upper Asymptotic Density*, Journal für die reine und angewandte Mathematik (Crelle's Journal), 595 (2006), 121 – 166.

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The Shnirel'man density σ of A is defined by

$$\sigma(A) = \inf_{n \geqslant 1} |A \cap [1, n]|/n.$$

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Existing theorems

Theorem (P. Erdős, 1937)

Let B be a basis of order h. Then for any $A \subseteq \mathbb{N}_0$,

$$\sigma(A+B) \geqslant \sigma(A) + \frac{1}{2h}\sigma(A)(1-\sigma(A)).$$

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Buy-one-get-one-free scheme: One can derive a theorem about upper Banach density for each existing theorem about Shnirel'man density or lower asymptotic density.

Proposition (R. J., 2001)

Let $A \subseteq \mathbb{N}$. Then $\overline{BD}(A) \geqslant \alpha$ if and only if there is an $a \in {}^*\mathbb{N}$ such that

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Applications

Let $B \subseteq \mathbb{N}_0$. The set B is an upper (lower) asymptotic basis of order h if $\overline{d}(hB) = 1$ ($\underline{d}(hB) = 1$). The set B is an upper (lower) Banach density of order h if $\overline{BD}(hB) = 1$ ($\underline{BD}(hB) = 1$).

Theorem (R. J., 2011)

If B is a lower asymptotic basis of order h, then for any $A \subseteq \mathbb{N}$

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Corollary

Let P be the set of all prime numbers and Q be the set of all cubes of non-negative integers. Then for any $A \subseteq \mathbb{N}$,

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