Forcing Axioms and Their Applications

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Abstract

Forcing Axioms have been investigated in depth in axiomatic set theory as well as in set theoretic topology for decades. In the first part of this talk, we will focus on the first forcing axiom, Martin's Axiom, by studying its typical applications. In the second part, we will introduce various forms of forcing axioms and briefly review their properties, connections and applications.

1 Martin's Axiom
- Formulation
- Applications on Cardinal Invariants

2 Forcing Axioms
- PFA and MM
- Applications
- Variations

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A Suslin tree is an ω_1-tree with only countable chains and antichains.

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Martin and Solovay isolated a principle as a "short cut":

Theorem ([Martin and Solovay, 1970])

$\mathrm{MA}_{\aleph_1}$ *implies that there is no Suslin tree.*

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MA_κ: *if* $(\mathbb{P}, <)$ *satisfies countable chain condition (c.c.c.), and if* $\mathcal{D}$ *is a collection of dense subsets of* $\mathbb{P}$ *with* $|\mathcal{D}| \leq \kappa$, *then there exists a* $\mathcal{D}$*-generic filter of* $\mathbb{P}$.

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- $G \subseteq \mathbb{P}$ is $\mathcal{D}$-generic filter, if $\forall D \in \mathcal{D}$, $G \cap D \neq \emptyset$.

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Typical c.c.c. forcings include Cohen forcing, Random forcing, Suslin tree forcing, etc.

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Hence $\mathfrak{d} > \kappa$, we conclude that $\mathfrak{d} = \mathfrak{c}$.

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- *Any countable set of reals is null.*
- *There is a null set which has cardinality $\mathfrak{c}$.*
- $\text{non}(\mathcal{L})$*, is the least cardinality of some $A \subseteq \mathbb{R}$, such that $A \notin \mathcal{L}$.*
- $\aleph_1 \leq \text{non}(\mathcal{L}) \leq \mathfrak{c}$.

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Proof: Assume $\kappa < \mathfrak{c}$ and $Y = \{y_\alpha | \alpha < \kappa\}$. We show that $Y \in \mathcal{L}$.
$\forall \epsilon > 0$, find an open set g with $\mu(g) \leq \epsilon$ and $Y \subseteq g$.

- $\mathbb{P}$: a condition p is a union of open intervals with rational endpoints and $\mu(p) < \epsilon$; $p \leq q$ iff $p \supseteq q$.
- $\mathbb{P}$ is countable.
- For each α, $D_\alpha = \{p | y_\alpha \in p\}$ is dense.
- By Martin's Axiom, let G be generic over these dense sets, and let $g = \bigcup G$.
- Then $Y \subseteq g$ and g is an open set.
- Key: $\mu(g) \leq \epsilon$.

Hence $\mathrm{non}(\mathcal{L}) > \kappa$, we conclude that $\mathrm{non}(\mathcal{L}) = \mathfrak{c}$.

Covering number for null sets

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Random forcing

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Proof:If $X \subseteq \mathbb{P}$ is an antichain, then $\forall p, q \in X$, either $p = q$, or $p \cap q = \emptyset$.
Let $X_n = \{p \in X : \mu(p) > 1/n\}$. Since the measure of $(0,1)$ is 1, X_n has less than n elements. Therefore, $X = \bigcup_{n<\omega} X_n$ is countable.

Random real

- $E_n = \{p | \exists m < 2^n, p \subseteq (\frac{m}{2^n}, \frac{m+1}{2^n})\}$ is dense.

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- $r_g \in \bigcap G$ is called the random real.

Martin's axiom and covering number for null sets

To prove $MA \vdash \mathrm{cov}(\mathcal{L}) = \mathfrak{c}$, assume $\kappa < \mathfrak{c}$ and $\mathcal{A} = \{A_\alpha | \alpha < \kappa\} \subseteq \mathcal{L}$. We use random forcing to show that $\bigcup \mathcal{A} \not\supseteq (0, 1)$.

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- $G \cap D_\alpha \neq \emptyset$ guarantees that $r_g \notin A_\alpha$.

Hence $g \notin \bigcup \mathcal{A}$ and $\text{cov}(\mathcal{L}) > \kappa$, we conclude that $\text{cov}(\mathcal{L}) = \mathfrak{c}$.

1. Martin's Axiom
   - Formulation
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2. Forcing Axioms
   - PFA and MM
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Proper forcing axiom

- Baumgartner [Baumgartner, 1983] generalized the property of countable chain condition to “Axiom A”, which includes typical forcings like Sacks forcing, Mathias forcing and Laver forcing.

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- Shelah [Shelah, 1982] soon made a great step further by developing the notion of “properness”.
- $\mathbb{P}$ is proper, if for every $\kappa \geq \omega_1$, every stationary $S \subseteq [\kappa]^\omega$, S is stationary in $\mathbb{V}^{\mathbb{P}}$.

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countable chain condition	Axiom A	properness
finite support	finite iteration	countable support
preserve cardinals	preserve $\aleph_1$	preserve $\aleph_1$

Proper Forcing Axiom

Definition ([Baumgartner, 1984])

Proper Forcing Axiom (PFA)*: if* $(\mathbb{P}, <)$ *is proper, and if* $\mathcal{D}$ *is a collection of dense subsets of* $\mathbb{P}$ *with* $|\mathcal{D}| = \aleph_1$*, then there exists a* $\mathcal{D}$*-generic filter of* $\mathbb{P}$*.*

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Theorem ([Baumgartner, 1984])

PFA *implies* $\neg\mathrm{MA}_{\aleph_2}$*.*

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A proper forcing preserves not only ω_1, but also the stationarity of subsets of ω_1. Indeed, this is the least requirement for a similar principle:

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Clearly, MM implies PFA.

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Applications: Cardinal arithmetic

Theorem ([Martin and Solovay, 1970])

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PFA *implies that* $\mathfrak{c} = \aleph_2$.

Theorem ([Viale, 2006])

PFA *proves the singular cardinal hypothesis.*

Namely: for every singular strong limit cardinal κ, $2^\kappa = \kappa^+$.

Applications: Continuum and linear orders

Theorem ([Martin and Solovay, 1970])

Assume MA, *then* $\mathrm{add}(\mathcal{L}) = \mathfrak{c}$.

Namely, the union of $< \mathfrak{c}$ many Lebesgue measure zero sets has Lebesgue measure zero.

Applications: Continuum and linear orders

Theorem ([Martin and Solovay, 1970])

Assume MA, *then* $\text{add}(\mathcal{L}) = \mathfrak{c}$.

Theorem ([Baumgartner, 1984])

Assume PFA, *then all* $\aleph_1$*-dense sets of reals are order isomorphic.*

$X \subseteq \mathbb{R}$ is $\aleph_1$-dense, if $\forall x < y$, there are exactly $\aleph_1$ many reals of X lie in the inteval (x, y).

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Theorem ([Moore, 2006])

Assume PFA, *then there is a five elements basis for uncountable linear orders.*

Any $\aleph_1$-dense $X \subseteq \mathbb{R}$, $(\omega_1, \in)$, $(\omega_1, \ni)$, a Countryman line C and its reverse C^*.

Applications: Trees

Theorem ([Baumgartner et al., 1970])

Assume $\mathrm{MA}_{\aleph_1}$*, then every Aronszajn tree is special.*

An Aronszajn tree T is an ω_1-tree with only countable chains and levels. T is special, if there is an order preserving mapping from T into the rationals.

Applications: Trees

Theorem ([Baumgartner et al., 1970])

Assume $\mathrm{MA}_{\aleph_1}$, *then every Aronszajn tree is special.*

Theorem ([Abraham and Shelah, 1985])

Assume PFA, *then all* $\aleph_1$*-Aronszajn trees are club isomorphic.*

T and T' are club isomorphic, if there is a closed and unbounded $C \subseteq \omega_1$, such that $T \restriction_C = T' \restriction_C$.

Applications: Higher structures

Theorem ([Todorčević, 1984])

Assume PFA*, then for every uncountable cardinal* κ*,* $\square_\kappa$ *fails.*

$\square_\kappa$: there is a sequence $\langle C_\alpha : \alpha \in Lim(\kappa^+), \kappa < \alpha < \kappa^+ \rangle$, with each C_α a club of α, $otp(C_\alpha) \leq \kappa$ for $cf(\alpha) < \kappa$, and $C_\alpha = C_\beta \cap \alpha$ for $\alpha \in Lim(C_\beta)$.

Applications: Higher structures

Theorem ([Todorčević, 1984])

Assume PFA, *then for every uncountable cardinal* κ, $\square_\kappa$ *fails.*

Theorem ([Foreman et al., 1988])

Assume MM, *then the nonstationary ideal on* ω_1 *is* $\aleph_2$*-saturated.*

Namely, the boolean algebra $\mathcal{P}(\omega_1)\ /I$ has $\aleph_2$-c.c.

Applications: Popular principles

Theorem ([Todorčević, 1989])

Assume PFA*, then the open coloring axiom* (OCA) *holds.*

Theorem ([Abraham and Todorčević, 1997], [Todorčević, 2000])

Assume PFA*, then the* P*-ideal dichotomy* (PID) *holds.*

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Theorem ([Moore, 2005])

Assume PFA, *then the mapping reflection principle* (MRP) *holds.*

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Forcing axioms in general

Definition

If Γ is a class of partial orders, κ is a cardinal:

- $\mathrm{FA}_\kappa(\Gamma)$: *For every $\mathbb{P} \in \Gamma$, and if $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, then there exists a $\mathcal{D}$-generic filter of $\mathbb{P}$.*
- $\mathrm{FA}_{<\mathfrak{c}}(\Gamma)$: *For every $\mathbb{P} \in \Gamma$, and if $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \mathfrak{c}$, then there exists a $\mathcal{D}$-generic filter of $\mathbb{P}$.*

Forcing axioms in general

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If Γ is a class of partial orders, κ is a cardinal:

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Fact

- MA_κ *is* FA_κ*(c.c.c.);*
- PFA *is* $\mathrm{FA}_{\aleph_1}$ *(proper);*
- MM *is* $\mathrm{FA}_{\aleph_1}$ *(stationary subsets of ω_1 preserving).*

Variations of MA (when $\kappa < \mathfrak{c}$)

$$\begin{array}{ccc}
\mathrm{MA} & \longrightarrow & \mathrm{MA}_\kappa \\
\downarrow & & \downarrow \\
\mathrm{FA}_{<\mathfrak{c}}(\textit{Knaster's condition}) & \longrightarrow & \mathrm{FA}_\kappa(\textit{Knaster's condition}) \\
\downarrow & & \downarrow \\
\mathrm{FA}_{<\mathfrak{c}}(\sigma - \textit{linked}) & \longrightarrow & \mathrm{FA}_\kappa(\sigma - \textit{linked}) \\
\downarrow & & \downarrow \\
\mathrm{FA}_{<\mathfrak{c}}(\sigma - \textit{centered}) & \longrightarrow & \mathrm{FA}_\kappa(\sigma - \textit{centered})
\end{array}$$

Bounded forcing axioms

- $\mathrm{BFA}_\kappa(\Gamma)$: For every $\mathbb{P} \in \Gamma$, and if $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$ and for each $D \in \mathcal{D}$, $|D| \leq \kappa$, then there exists a $\mathcal{D}$-generic filter of $\mathbb{P}$.

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- $\mathsf{FA}_{\kappa,\lambda}(\Gamma)$: For every $\mathbb{P} \in \Gamma$, and if $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$ and for each $D \in \mathcal{D}$, $|D| \leq \lambda$, then there exists a $\mathcal{D}$-generic filter of $\mathbb{P}$.

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- BMM is $\mathrm{BFA}_{\aleph_1}$(stationary subsets of ω_1 preserving).

Theorem ([Bagaria, 2000])

The following are equivalent:

- $\mathrm{BFA}_\kappa(\mathbb{P})$
- $(\mathcal{P}(\kappa), \in) \prec_{\Sigma_1} (\mathbb{V}^{\mathbb{P}}, \in)$
- $(H_{\kappa^+}, \in) \prec_{\Sigma_1} (\mathbb{V}^{\mathbb{P}}, \in)$

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  The property of “no new reals” is not preserved by countable support iteration;

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- *w*PFA ([Bagaria et al., 2017]).

A comparison of variations of PFA:

FA	imply	not imply
BPFA	MA_{ω_1}, $\mathfrak{c} = \aleph_2$	OCA, MRP, PID
PFA(S)	OCA, $\mathfrak{c} = \aleph_2$, $\neg\square_\kappa$	SH,PID
$< \omega_1 -$ PFA	OCA, PID, $\mathfrak{c} = \aleph_2$	$\neg$CG(ω_1)
DPFA	PID, MRP	OCA, MA_{ω_1}

Subcomplete forcing axiom

- Subcomplete forcings [Jensen, 2009], [Jensen, 2014] do not add reals, preserve stationarity of subsets of ω_1, are closed under revised countable support iteration, but may change cofinality to ω.

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- Examples: countably closed forcing, Namba forcing (under CH), Prikry forcing, etc.

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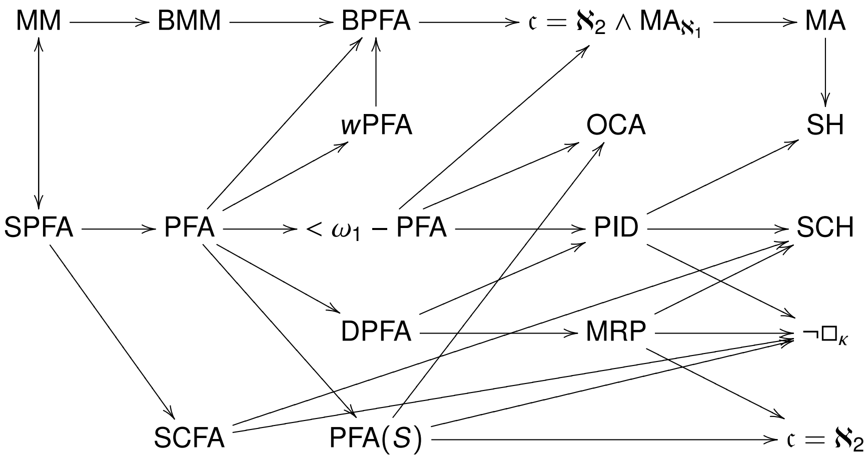
- SCFA is consistent with CH, or even $\diamondsuit$.

Large cardinal strength (to be updated)

FA	an upper bound	a lower bound
MA	ω	ω
PFA	supercompact	ω many Woodins
MM	supercompact	ω many Woodins
BPFA	Σ_1-reflecting	Σ_1-reflecting
BMM	$\omega + 1$ many Woodins	$\forall X, X^{\#}$ exists
$< \omega_1 -$ PFA	supercompact	ω many Woodins
DPFA	supercompact	ω many Woodins
PFA(S)	supercompact	ω many Woodins
wPFA	remarkable	remarkable
SCFA	supercompact	ω many Woodins

Connections



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Thanks for attention!