Abstract

Forcing Axioms have been investigated in depth in axiomatic set theory as well as in set theoretic topology for decades. In the first part of this talk, we will focus on the first forcing axiom, Martin’s Axiom, by studying its typical applications. In the second part, we will introduce various forms of forcing axioms and briefly review their properties, connections and applications.
1. **Martin’s Axiom**
   - Formulation
   - Applications on Cardinal Invariants

2. **Forcing Axioms**
   - PFA and MM
   - Applications
   - Variations
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Birth of Martin’s axiom

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A Suslin tree is an $\omega_1$-tree with only countable chains and antichains.
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**Theorem ([Martin and Solovay, 1970])**

\[ \text{MA}_{\aleph_1} \text{ implies that there is no Suslin tree.} \]
Slices of Martin’s axiom

Definition ([Martin and Solovay, 1970])

\( \text{MA}_\kappa: \) if \( (\mathbb{P}, <) \) satisfies countable chain condition (c.c.c.), and if \( \mathcal{D} \) is a collection of dense subsets of \( \mathbb{P} \) with \( |\mathcal{D}| \leq \kappa \), then there exists a \( \mathcal{D} \)-generic filter of \( \mathbb{P} \).
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\[ \text{MA: if } (\mathbb{P}, <) \text{ satisfies countable chain condition (c.c.c.), and if } \mathcal{D} \text{ is a collection of dense subsets of } \mathbb{P} \text{ with } |\mathcal{D}| < \mathfrak{c}, \text{ then there exists a } \mathcal{D}\text{-generic filter of } \mathbb{P}. \]
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**Definition ([Martin and Solovay, 1970])**

$\text{MA}_\kappa$: if $(\mathbb{P}, <)$ satisfies countable chain condition (c.c.c.), and if $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, then there exists a $\mathcal{D}$-generic filter of $\mathbb{P}$.

**Definition ([Martin and Solovay, 1970])**

$\text{MA}$: if $(\mathbb{P}, <)$ satisfies countable chain condition (c.c.c.), and if $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < c$, then there exists a $\mathcal{D}$-generic filter of $\mathbb{P}$. 
Notations

- Forking requirement: \( \forall p \in P \ \exists q \leq p \ \exists r \leq p \) such that there is no \( s \), with \( s \leq q \) and \( s \leq r \).
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- \( \mathfrak{c} \) is the continuum, the cardinality of the collection of reals.
- \( \emptyset \neq G \subseteq P \) is a filter, if \( (p \in G \land p \leq q) \rightarrow q \in G \) and \( p, q \in G \rightarrow \exists r \in G(r \leq p \land r \leq q) \).
Forcing Axioms and Their Applications

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- $G \subseteq \mathbb{P}$ is $\mathcal{D}$-generic filter, if $\forall D \in \mathcal{D}, G \cap D \neq \emptyset$. 
Easy facts

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- $\text{MA}_{\aleph_0}$ is true, $\text{MA}_c$ is false.
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- $\text{MA}_\omega$ is true, $\text{MA}_c$ is false.
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Typical c.c.c. forcings include Cohen forcing, Random forcing, Suslin tree forcing, etc.
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   - PFA and MM
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The dominating number

**Definition**

For \( f, g \in \omega^\omega \), \( f \) is dominated by \( g \), denoted by \( f \leq^* g \), if 
\[ \exists m \in \omega \forall n \geq m \ f(n) \leq g(n). \]
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  \[ \exists m \in \omega \forall n \geq m \ f(n) \leq g(n). \]
- $\mathcal{F} \subseteq \omega^\omega$ is a dominating family, if $\forall f \in \omega^\omega \exists g \in \mathcal{F} \ f \leq^* g.$
Forcing Axioms and Their Applications

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- The dominating number, denoted by $\mathfrak{d}$, is the least cardinality of a dominating family.
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- $\aleph_1 \leq d \leq \mathfrak{c}$.
Martin’s axiom and the dominating number

Theorem ([Martin and Solovay, 1970])

*Martin’s Axiom implies* $\mathfrak{d} = \mathfrak{c}$. 
Forcing Axioms and Their Applications

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Proof: Assume $\kappa < c$, $F = \{ f_\alpha | \alpha < \kappa \} \subseteq \omega^\omega$, we will find some $g \in \omega^\omega$ such that $g$ is not dominated by any function in $F$. 

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- Let $g = \bigcup G$. Then $g \in \omega^\omega$, and $g$ is not dominated by any $f_\alpha$. 
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- Let \( g = \bigcup G \). Then \( g \in \omega^\omega \), and \( g \) is not dominated by any \( f_\alpha \).

Hence \( d > \kappa \), we conclude that \( d = c \).
The refining number

**Definition**

- Suppose $A, B \in [\omega]^\omega$, say $B$ splits $A$, if both $A \cap B$ and $A \setminus B$ are infinite.
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- For $\alpha < \kappa$, $m < \omega$, $D_{\alpha,m} = \{p : |A_\alpha \cap p| > m\}$, $E_{\alpha,m} = \{p : |(A_\alpha \cap \max(p)) \setminus p| > m\}$ are dense.
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- By Martin’s Axiom, let \( G \) be generic over these dense sets, and let \( g = \bigcup G \).
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Definition

- Let $\mathcal{L}$ denote the collection of null sets, or sets of reals which have Lebesgue measure zero.
- Any countable set of reals is null.
- There is a null set which has cardinality $\mathfrak{c}$.
- $\text{non}(\mathcal{L})$, is the least cardinality of some $A \subseteq \mathbb{R}$, such that $A \notin \mathcal{L}$.
- $\aleph_1 \leq \text{non}(\mathcal{L}) \leq \mathfrak{c}$. 
Martin’s axiom and the non-null number

Theorem ([Martin and Solovay, 1970])

Martin’s Axiom implies \( \text{non}(\mathcal{L}) = c \).
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Proof: Assume \( \kappa < \mathfrak{c} \) and \( Y = \{ y_\alpha | \alpha < \kappa \} \). We show that \( Y \in \mathcal{L} \).

\( \forall \epsilon > 0 \), find an open set \( g \) with \( \mu(g) \leq \epsilon \) and \( Y \subseteq g \).
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\( \mathbb{P} \): a condition \( p \) is a union of open intervals with rational endpoints and \( \mu(p) < \epsilon \); \( p \leq q \) iff \( p \supseteq q \).
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- $\mathbb{P}$: a condition $p$ is a union of open intervals with rational endpoints and $\mu(p) < \epsilon$; $p \leq q$ iff $p \supseteq q$.
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- For each \( \alpha \), \( D_\alpha = \{ p | y_\alpha \in p \} \) is dense.
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Proof: Assume $\kappa < \mathfrak{c}$ and $Y = \{y_\alpha | \alpha < \kappa\}$. We show that $Y \in \mathcal{L}$. For all $\epsilon > 0$, find an open set $g$ with $\mu(g) \leq \epsilon$ and $Y \subseteq g$.

- $\mathcal{P}$: A condition $p$ is a union of open intervals with rational endpoints and $\mu(p) < \epsilon$; $p \leq q$ if $p \supseteq q$.
- $\mathcal{P}$ is countable.
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- By Martin’s Axiom, let $G$ be generic over these dense sets, and let $g = \bigcup G$. 
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$\forall \epsilon > 0$, find an open set $g$ with $\mu(g) \leq \epsilon$ and $Y \subseteq g$.

- $\mathbb{P}$: a condition $p$ is a union of open intervals with rational endpoints and $\mu(p) < \epsilon$; $p \leq q$ iff $p \supseteq q$.
- $\mathbb{P}$ is countable.
- For each $\alpha$, $D_\alpha = \{p | y_\alpha \in p\}$ is dense.
- By Martin’s Axiom, let $G$ be generic over these dense sets, and let $g = \bigcup G$.
- Then $Y \subseteq g$ and $g$ is an open set.
Martin’s axiom and the non-null number

Theorem ([Martin and Solovay, 1970])

**Martin’s Axiom implies** $\text{non}(\mathcal{L}) = \mathfrak{c}$.

Proof: Assume $\kappa < \mathfrak{c}$ and $Y = \{y_\alpha | \alpha < \kappa\}$. We show that $Y \in \mathcal{L}$. 

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- Then $Y \subseteq g$ and $g$ is an open set.

- Key: $\mu(g) \leq \epsilon$.

Hence $\text{non}(\mathcal{L}) > \kappa$, we conclude that $\text{non}(\mathcal{L}) = \mathfrak{c}$. 
Covering number for null sets

**Definition**

- A *union of countably many null sets does not cover* $\mathbb{R}$.
Covering number for null sets

Definition

- A union of countably many null sets does not cover $\mathbb{R}$
- A covering family from $\mathcal{I}$ is some $\mathcal{X} \subseteq \mathcal{I}$, such that $\bigcup \mathcal{X} = \mathbb{R}$. 
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- A covering family from $\mathcal{I}$ is some $\mathcal{X} \subseteq \mathcal{I}$, such that $\bigcup \mathcal{X} = \mathbb{R}$.
- The covering number for $\mathcal{L}$, denoted by $\text{cov}(\mathcal{L})$, is the least cardinality of a family $\mathcal{X} \subseteq \mathcal{L}$, such that $\bigcup \mathcal{X} = \mathbb{R}$. 

Theorem ([Martin and Solovay, 1970]) Martin’s Axiom implies $\text{cov}(\mathcal{L}) = c$. 

\[ \aleph_1 \leq \text{cov}(\mathcal{L}) \leq c. \]
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Theorem ([Martin and Solovay, 1970])

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**Lemma**

*Random forcing satisfies countable chain condition.*

**Proof:** If \(X \subseteq \mathcal{P}\) is an antichain, then \(\forall p, q \in X\), either \(p = q\), or \(p \cap q = \emptyset\).

Let \(X_n = \{p \in X : \mu(p) > 1/n\}\). Since the measure of \((0, 1)\) is 1, \(X_n\) has less than \(n\) elements. Therefore, \(X = \bigcup_{n<\omega} X_n\) is countable.
Random real

\[ E_n = \{ p \mid \exists m < 2^n, p \subseteq (\frac{m}{2^n}, \frac{m+1}{2^n}) \} \text{ is dense.} \]
Random real

- \( E_n = \{ p | \exists m < 2^n, p \subseteq (\frac{m}{2^n}, \frac{m+1}{2^n}) \} \) is dense.

- If \( G \) is generic over each \( E_n \), then for each \( n \), there is a unique \( m = m(n) \) such that \( \bigcup (E_n \cap G) \subseteq (\frac{m}{2^n}, \frac{m+1}{2^n}) \).
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- $r_g \in \bigcap G$ is called the random real.
Martin’s axiom and covering number for null sets

To prove $MA \vdash \text{cov}(\mathcal{L}) = \mathfrak{c}$, assume $\kappa < \mathfrak{c}$ and $\mathcal{A} = \{ A_\alpha | \alpha < \kappa \} \subseteq \mathcal{L}$. We use random forcing to show that $\bigcup \mathcal{A} \not\in (0, 1)$. 

For each $\alpha < \kappa$, let $D_\alpha = \{ p | p \cap A_\alpha = \emptyset \}$. Given $q \in P$, $\alpha < \kappa$, let $B$ be a $\mathcal{G}_\delta$ set such that $B \in \mathcal{L}$ and $B \supseteq A_\alpha$. Let $p = q \setminus B$, then $p \leq q$ and $p \in D_\alpha$. So $D_\alpha$ is dense.

By Martin’s Axiom, let $G$ be generic over $D_\alpha$'s and $E_n$'s, and let $r_g$ be the random real decided by $G$. $G \cap D_\alpha$, $\emptyset$ guarantees that $r_g < A_\alpha$. Hence $g < \bigcup \mathcal{A}$ and $\text{cov}(\mathcal{L}) > \kappa$, we conclude that $\text{cov}(\mathcal{L}) = \mathfrak{c}$. 
To prove $MA \vdash \text{cov}(L) = \mathfrak{c}$, assume $\kappa < \mathfrak{c}$ and $\mathcal{A} = \{A_\alpha | \alpha < \kappa\} \subseteq L$. We use random forcing to show that $\bigcup \mathcal{A} \not\in (0, 1)$.

- For each $\alpha < \kappa$, let $D_\alpha = \{p | p \cap A_\alpha = \emptyset\}$. 
Martin’s axiom and covering number for null sets

To prove $MA \vdash \text{cov}(\mathcal{L}) = \omega$, assume $\kappa < \omega$ and $\mathcal{A} = \{A_\alpha | \alpha < \kappa\} \subseteq \mathcal{L}$. We use random forcing to show that $\bigcup \mathcal{A} \notin (0, 1)$.

- For each $\alpha < \kappa$, let $D_\alpha = \{p | p \cap A_\alpha = \emptyset\}$.
- Given $q \in \mathbb{P}, \alpha < \kappa$, let $B$ be a $G_\delta$ set such that $B \in \mathcal{L}$ and $B \supseteq A_\alpha$. Let $p = q \setminus B$, then $p \leq q$ and $p \in D_\alpha$. So $D_\alpha$ is dense.
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To prove $\mathsf{MA} \vdash \text{cov}(\mathcal{L}) = \mathfrak{c}$, assume $\kappa < \mathfrak{c}$ and $\mathcal{A} = \{A_\alpha | \alpha < \kappa\} \subseteq \mathcal{L}$. We use random forcing to show that $\bigcup \mathcal{A} \not\in (0, 1)$.

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Hence $g \notin \bigcup \mathcal{A}$ and $\text{cov}(\mathcal{L}) > \kappa$, we conclude that $\text{cov}(\mathcal{L}) = \mathfrak{c}$. 
Forcing Axioms and Their Applications

1. **Martin’s Axiom**
   - Formulation
   - Applications on Cardinal Invariants

2. **Forcing Axioms**
   - PFA and MM
   - Applications
   - Variations
Baumgartner [Baumgartner, 1983] generalized the property of countable chain condition to “Axiom A”, which includes typical forcings like Sacks forcing, Mathias forcing and Laver forcing.
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$\mathbb{P}$ is proper, if for every $\kappa \geq \omega_1$, every stationary $S \subseteq [\kappa]^\omega$, $S$ is stationary in $\forall^\mathbb{P}$. 
Baumgartner [Baumgartner, 1983] generalized the property of countable chain condition to “Axiom A”, which includes typical forcings like Sacks forcing, Mathias forcing and Laver forcing.

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Proper Forcing Axiom

**Definition ([Baumgartner, 1984])**

*Proper Forcing Axiom (PFA):* if \((\mathbb{P}, \prec)\) is proper, and if \(\mathcal{D}\) is a collection of dense subsets of \(\mathbb{P}\) with \(|\mathcal{D}| = \aleph_1\), then there exists a \(\mathcal{D}\)-generic filter of \(\mathbb{P}\).
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Clearly, PFA $\rightarrow$ MA$_{\aleph_1}$. Moreover,
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Theorem ([Baumgartner, 1984])

PFA implies $\neg$MA$_{\aleph_2}$. 
Martin’s Maximum

A proper forcing preserves not only $\omega_1$, but also the stationarity of subsets of $\omega_1$. Indeed, this is the least requirement for a similar principle:

Definition ([Foreman et al., 1988]) Martin’s Maximum (MM): if $(P, <)$ preserves the stationary subsets of $\omega_1$, and if $D$ is a collection of dense subsets of $P$ with $|D| = \aleph_1$, then there exists a $D$-generic filter of $P$. Clearly, MM implies PFA.
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1 Martin’s Axiom
   - Formulation
   - Applications on Cardinal Invariants

2 Forcing Axioms
   - PFA and MM
   - Applications
   - Variations
Applications: Cardinal arithmetic

Theorem ([Martin and Solovay, 1970])

MA implies that \( c \) is regular, and \( \forall \kappa < c, 2^\kappa = c \).
Applications: Cardinal arithmetic

Theorem ([Martin and Solovay, 1970])

MA implies that $c$ is regular, and $\forall \kappa < c, 2^\kappa = c$.

Theorem ([Todorčević, 1989], [Veličković, 1992])

PFA implies that $c = \aleph_2$. 
Forcing Axioms and Their Applications

Applications: Cardinal arithmetic

Theorem ([Martin and Solovay, 1970])

MA implies that $\mathfrak{c}$ is regular, and $\forall \kappa < \mathfrak{c}, 2^\kappa = \mathfrak{c}$.

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PFA implies that $\mathfrak{c} = \aleph_2$.

So PFA implies MA.
Applications: Cardinal arithmetic

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MA implies that $\mathfrak{c}$ is regular, and $\forall \kappa < \mathfrak{c}, 2^\kappa = \mathfrak{c}$.

Theorem ([Todorčević, 1989], [Veličković, 1992])

PFA implies that $\mathfrak{c} = \aleph_2$.

Theorem ([Viale, 2006])

PFA proves the singular cardinal hypothesis.

Namely: for every singular strong limit cardinal $\kappa$, $2^\kappa = \kappa^+$. 
Theorem ([Martin and Solovay, 1970])

Assume MA, then \( \text{add}(\mathcal{L}) = \aleph_1 \).

Namely, the union of \( < \aleph_1 \) many Lebesgue measure zero sets has Lebesgue measure zero.
Applications: Continuum and linear orders

Theorem ([Martin and Solovay, 1970])
Assume MA, then \( \text{add}(\mathcal{L}) = \mathfrak{c} \).

Theorem ([Baumgartner, 1984])
Assume PFA, then all \( \aleph_1 \)-dense sets of reals are order isomorphic.

\( X \subseteq \mathbb{R} \) is \( \aleph_1 \)-dense, if \( \forall x < y \), there are exactly \( \aleph_1 \) many reals of \( X \) lie in the interval \( (x, y) \).
Applications: Continuum and linear orders

Theorem ([Martin and Solovay, 1970])
Assume MA, then $\text{add}(\mathcal{L}) = \mathfrak{c}$.

Theorem ([Baumgartner, 1984])
Assume PFA, then all $\mathfrak{c}_1$-dense sets of reals are order isomorphic.

Theorem ([Moore, 2006])
Assume PFA, then there is a five elements basis for uncountable linear orders.

Any $\mathfrak{c}_1$-dense $X \subseteq \mathbb{R}$, $(\omega_1, \epsilon)$, $(\omega_1, \varpi)$, a Countryman line $C$ and its reverse $C^*$.
Applications: Trees

Theorem ([Baumgartner et al., 1970])
Assume MA$_{\aleph_1}$, then every Aronszajn tree is special.

An Aronszajn tree $T$ is an $\omega_1$-tree with only countable chains and levels. $T$ is special, if there is an order preserving mapping from $T$ into the rationals.
Applications: Trees

Theorem ([Baumgartner et al., 1970])

Assume $\text{MA}_{\aleph_1}$, then every Aronszajn tree is special.

Theorem ([Abraham and Shelah, 1985])

Assume PFA, then all $\aleph_1$-Aronszajn trees are club isomorphic.

$T$ and $T'$ are club isomorphic, if there is a closed and unbounded $C \subseteq \omega_1$, such that $T \upharpoonright C = T' \upharpoonright C$. 
Forcing Axioms and Their Applications

Applications: Higher structures

Theorem ([Todorčević, 1984])

Assume PFA, then for every uncountable cardinal $\kappa$, $\square_\kappa$ fails.

$\square_\kappa$: there is a sequence $\langle C_\alpha : \alpha \in \text{Lim}(\kappa^+) \rangle$, $\kappa < \alpha < \kappa^+$, with each $C_\alpha$ a club of $\alpha$, $\text{otp}(C_\alpha) \leq \kappa$ for $\text{cf}(\alpha) < \kappa$, and $C_\alpha = C_\beta \cap \alpha$ for $\alpha \in \text{Lim}(C_\beta)$. 
Theorem ([Todorčević, 1984])

Assume PFA, then for every uncountable cardinal $\kappa$, $\square_\kappa$ fails.

Theorem ([Foreman et al., 1988])

Assume MM, then the nonstationary ideal on $\omega_1$ is $\mathfrak{S}_2$-saturated.

Namely, the boolean algebra $\mathcal{P}(\omega_1) / I$ has $\mathfrak{S}_2$-c.c.
Applications: Popular principles

Theorem ([Todorčević, 1989])

Assume PFA, then the open coloring axiom (OCA) holds.

Theorem ([Abraham and Todorčević, 1997], [Todorčević, 2000])

Assume PFA, then the P-ideal dichotomy (PID) holds.
### Applications: Popular principles

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   - Applications on Cardinal Invariants

2. **Forcing Axioms**
   - PFA and MM
   - Applications
   - Variations
Forcing axioms in general

Definition

If $\Gamma$ is a class of partial orders, $\kappa$ is a cardinal:

- $\text{FA}_\kappa(\Gamma)$: For every $P \in \Gamma$, and if $D$ is a collection of dense subsets of $P$ with $|D| \leq \kappa$, then there exists a $D$-generic filter of $P$.

- $\text{FA}_{<\kappa}(\Gamma)$: For every $P \in \Gamma$, and if $D$ is a collection of dense subsets of $P$ with $|D| < \kappa$, then there exists a $D$-generic filter of $P$. 

Fact $\text{MA}_\kappa$ is $\text{FA}_\kappa(\text{c.c.c.})$; $\text{PFA}$ is $\text{FA}_{\aleph_1}$ (proper); $\text{MM}$ is $\text{FA}_{\aleph_1}$ (stationary subsets of $\omega_1$ preserving).
Forcing axioms in general

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- $\text{FA}_{<\mathfrak{c}}(\Gamma)$: For every $P \in \Gamma$, and if $D$ is a collection of dense subsets of $P$ with $|D| < \mathfrak{c}$, then there exists a $D$-generic filter of $P$.

Fact

- $\text{MA}_\kappa$ is $\text{FA}_\kappa(\text{c.c.c.})$;
- $\text{PFA}$ is $\text{FA}_{\aleph_1}$ (proper);
- $\text{MM}$ is $\text{FA}_{\aleph_1}$ (stationary subsets of $\omega_1$ preserving).
Variations of MA (when $\kappa < \mathfrak{c}$)

\[ \text{MA} \xrightarrow{} \text{MA}_\kappa \]

\[ \text{FA}_{< \mathfrak{c}}(\text{Knaster's condition}) \xrightarrow{} \text{FA}_\kappa(\text{Knaster's condition}) \]

\[ \text{FA}_{< \mathfrak{c}}(\sigma - \text{linked}) \xrightarrow{} \text{FA}_\kappa(\sigma - \text{linked}) \]

\[ \text{FA}_{< \mathfrak{c}}(\sigma - \text{centered}) \xrightarrow{} \text{FA}_\kappa(\sigma - \text{centered}) \]
Bounded forcing axioms

- BFA$_\kappa(\Gamma)$: For every $P \in \Gamma$, and if $\mathcal{D}$ is a collection of dense subsets of $P$ such that $|\mathcal{D}| \leq \kappa$ and for each $D \in \mathcal{D}$, $|D| \leq \kappa$, then there exists a $\mathcal{D}$-generic filter of $P$. 
Bounded forcing axioms

- \( \text{BFA}_\kappa(\Gamma) \): For every \( \mathbb{P} \in \Gamma \), and if \( \mathcal{D} \) is a collection of dense subsets of \( \mathbb{P} \) such that \( |\mathcal{D}| \leq \kappa \) and for each \( D \in \mathcal{D} \), \( |D| \leq \kappa \), then there exists a \( \mathcal{D} \)-generic filter of \( \mathbb{P} \).

- \( \text{FA}_{\kappa,\lambda}(\Gamma) \): For every \( \mathbb{P} \in \Gamma \), and if \( \mathcal{D} \) is a collection of dense subsets of \( \mathbb{P} \) such that \( |\mathcal{D}| \leq \kappa \) and for each \( D \in \mathcal{D} \), \( |D| \leq \lambda \), then there exists a \( \mathcal{D} \)-generic filter of \( \mathbb{P} \).
Forcing Axioms and Their Applications

- Forcing Axioms
  - Variations

Bounded forcing axioms

- **BFA\(_\kappa(\Gamma)\):** For every \(\mathbb{P} \in \Gamma\), and if \(\mathcal{D}\) is a collection of dense subsets of \(\mathbb{P}\) such that \(|\mathcal{D}| \leq \kappa\) and for each \(D \in \mathcal{D}\), \(|D| \leq \kappa\), then there exists a \(\mathcal{D}\)-generic filter of \(\mathbb{P}\).

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- **BFA\(_{\kappa}(\Gamma)\) is FA\(_{\kappa,\kappa}(\Gamma)\).**
Bounded forcing axioms

- $\text{BFA}_\kappa(c.c.c.)$ is equivalent to $\text{MA}_\kappa$;
Bounded forcing axioms

- $\text{BFA}_\kappa(c.c.c.)$ is equivalent to $\text{MA}_\kappa$;
- $\text{BFA}_{\aleph_1}(\text{proper})$ is $\text{BPFA}$ [Goldstern and Shelah, 1995];
Bounded forcing axioms

- $\text{BFA}_\kappa (\text{c.c.c.})$ is equivalent to $\text{MA}_\kappa$;
- $\text{BFA}_{\aleph_1} (\text{proper})$ is BPFA [Goldstern and Shelah, 1995];
  BPFA is consistently weaker than PFA;
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- BMM is $\text{BFA}_{\aleph_1}$ (stationary subsets of $\omega_1$ preserving).

Theorem ([Bagaria, 2000])

*The following are equivalent:*

- $\text{BFA}_\kappa(\mathcal{P})$
- $(\mathcal{P}(\kappa), \in) <_{\Sigma_1} (\mathcal{V}^\mathcal{P}, \in)$
- $(H_{\kappa^+}, \in) <_{\Sigma_1} (\mathcal{V}^\mathcal{P}, \in)$
Other variations of PFA

- semiproper (Shelah): SPFA, equivalent to MM;
Other variations of PFA

- semiproper (Shelah): SPFA, equivalent to MM;
- $\alpha$-proper (Shelah): $< \omega_1$ – PFA;
Other variations of PFA

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Other variations of PFA

- semiproper (Shelah): SPFA, equivalent to MM;
- $\alpha$-proper (Shelah): $< \omega_1$ – PFA;
- preserving a coherent Suslin tree (Todorčević): PFA($S$);
- distributive and proper (Magidor): DPFA;
Other variations of PFA

- semiproper (Shelah): SPFA, equivalent to MM;
- $\alpha$-proper (Shelah): $< \omega_1$ – PFA;
- preserving a coherent Suslin tree (Todorčević): PFA(S);
- distributive and proper (Magidor): DPFA;
  The property of “no new reals” is not preserved by countable support iteration;
Other variations of PFA

- semiproper (Shelah): SPFA, equivalent to MM;
- $\alpha$-proper (Shelah): $< \omega_1$ – PFA;
- preserving a coherent Suslin tree (Todorčević): PFA(S);
- distributive and proper (Magidor): DPFA;
- $w$PFA ([Bagaria et al., 2017]).
A comparison of variations of PFA:

<table>
<thead>
<tr>
<th></th>
<th>imply</th>
<th>not imply</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPFA</td>
<td>$\text{MA}_{\omega_1}$, $c = \aleph_2$</td>
<td>$\text{OCA, MRP, PID}$</td>
</tr>
<tr>
<td>PFA(S)</td>
<td>$\text{OCA}$, $c = \aleph_2$, $\neg \square_k$</td>
<td>$\text{SH, PID}$</td>
</tr>
<tr>
<td>$&lt; \omega_1 - \text{PFA}$</td>
<td>$\text{OCA}$, $\text{PID}$, $c = \aleph_2$</td>
<td>$\neg \text{CG}(\omega_1)$</td>
</tr>
<tr>
<td>DPFA</td>
<td>$\text{PID}$, $\text{MRP}$</td>
<td>$\text{OCA}$, $\text{MA}_{\omega_1}$</td>
</tr>
</tbody>
</table>
Subcomplete forcing axiom

- Subcomplete forcings [Jensen, 2009], [Jensen, 2014] do not add reals, preserve stationarity of subsets of $\omega_1$, are closed under revised countable support iteration, but may change cofinality to $\omega$. 
Subcomplete forcing axiom

- Subcomplete forcings [Jensen, 2009], [Jensen, 2014] do not add reals, preserve stationarity of subsets of $\omega_1$, are closed under revised countable support iteration, but may change cofinality to $\omega$.

- Examples: countably closed forcing, Namba forcing (under CH), Prikry forcing, etc.
Subcomplete forcing axiom

SCFA is $\text{FA}_{\omega_1}(\text{subcomplete})$; [Jensen, 2009], [Jensen, 2014]
SCFA is $\text{FA}_{\aleph_1} (\text{subcomplete})$; [Jensen, 2009], [Jensen, 2014]

- MM implies SCFA;
- SCFA implies the SCH and $\neg \Box \kappa$, for $\kappa \geq \omega_1$;
Subcomplete forcing axiom

SCFA is $\text{FA}_{\aleph_1} (\text{subcomplete})$; [Jensen, 2009], [Jensen, 2014]

- MM implies SCFA;
- SCFA implies the SCH and $\neg \square_\kappa$, for $\kappa \geq \omega_1$;
- SCFA is consistent with CH, or even $\Diamond$. 
Large cardinal strength (to be updated)

<table>
<thead>
<tr>
<th></th>
<th>an upper bound</th>
<th>a lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MA</strong></td>
<td>( \omega )</td>
<td>( \omega )</td>
</tr>
<tr>
<td><strong>PFA</strong></td>
<td>supercompact</td>
<td>( \omega ) many Woodins</td>
</tr>
<tr>
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</tr>
<tr>
<td><strong>BPFA</strong></td>
<td>( \Sigma_1 )-reflecting</td>
<td>( \Sigma_1 )-reflecting</td>
</tr>
<tr>
<td><strong>BMM</strong></td>
<td>( \omega + 1 ) many Woodins</td>
<td>( \forall X, X# ) exists</td>
</tr>
<tr>
<td>( \omega_1 - \text{PFA} )</td>
<td>supercompact</td>
<td>( \omega ) many Woodins</td>
</tr>
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<td><strong>SCFA</strong></td>
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</tr>
</tbody>
</table>
Connections

\[
\begin{align*}
\text{MM} & \rightarrow \text{BMM} & \rightarrow \text{BPFA} & \rightarrow c = \aleph_2 \land \text{MA}_{\aleph_1} & \rightarrow \text{MA} \\
\text{SPFA} & \rightarrow \text{PFA} & \rightarrow < \omega_1 - \text{PFA} & \rightarrow \text{PID} & \rightarrow \text{SCH} \\
\text{SCFA} & \rightarrow \text{PFA}(S) & \rightarrow \text{DPFA} & \rightarrow \text{MRP} & \rightarrow \neg \Box_\kappa \\
\text{wPFA} & & & & \\
\text{OCA} & & & & \\
\text{SH} & & & & \\
\end{align*}
\]


References II

Iterated forcing.

Applications of the proper forcing axiom.

Martin’s maximum, saturated ideals, and non-regular ultrafilters. part i.
*Annals of Mathematics*, pages 1–47.

The bounded proper forcing axiom.

Subcomplete forcing and -forcing.

Forcing axioms compatible with ch. handwritten notes.
References III

References IV


Thanks for attention!