A survey on the computable Lipschitz reducibility

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Backgrounds

- Let M be a Turing machine that takes an input τ, views it as a natural number and outputs $M(\tau) = \sigma = 2^{2^{2^{\tau}}}$.
- For instance, if $\tau = 101$, then

$$M(\tau) = \sigma = 2^{2^{32}} = 2^{4294967296}$$

  and $|\sigma| = 2^{32}$.
- Given a Turing machine M, if $M(\tau) = \sigma$, then τ is an M-description of σ.

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- The Kolmogorov complexity of a string τ with respect to M via

$$C_M(\sigma) = min\{|\tau|, \infty \ : \ M(\tau) = \sigma\},$$

where $min\, \emptyset = \infty$.

- For a universal machine U,

$$C(\sigma) = C_U(\sigma) \leq C_M(\sigma) + O(1).$$

- The prefix-free Kolmogorov complexity: $K_M(\sigma), K(\sigma) = K_U(\sigma)$.

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Definition (Martin-Löf, P., 1966)

A real α is Martin-Löf random if for all uniformly c.e. sequences of open sets $\{U_n : n \in \omega\}$, with $\mu(U_n) \leq 2^{-n}, \alpha \notin \cap_n U_n$.

Theorem (Schnorr, 1973)

The following are equivalent for a real α.

1. α is *ML*-random;
2. $\exists c \forall n [K(\alpha \upharpoonright n) > n - c]$.

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- $\alpha \leq_E \beta$ if $E(\alpha \restriction n) \leq E(\beta \restriction n) + O(1)$, where $E = K$ or $E = C$.

- K-reducibility, C-reducibility, Solovay reducibility, computable Lipschitz reducibility, relative K-reducibility.

Theorem (Calude-Hertling-Khoussainov-Wang 2001; Kučera-Slaman 2001)

For a real β, the following are equivalent:

(1) β is left-c.e. and ML-random.

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(3) β is Solovay complete, that is, $\alpha \leq_S \beta$ for each left-c.e. real α.

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Computable Lipschitz reducibility

Definition (Downey,Hirschfeldt,2008)

Given two reals α and β, α is computable Lipschitz reducible to β ($\alpha \leq_{cl} \beta$)if there is a Turing functional Γ such that $\alpha = \Gamma^{\beta}$ and its use on n is bounded by $n + c$.

Proposition (Downey,Hirschfeldt and Lafort,2008)

If $\alpha \leq_{cl} \beta$, then for all n, $E(\alpha \upharpoonright n) \leq E(\beta \upharpoonright n) + O(1)$, here $E = K$ or $E = C$.

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Property (Downey, Hirschfeldt,Lafort 2001)

The cl-degrees of left-c.e. reals is neither a lower semi-lattice, nor an upper semi-lattice.

Theorem (Yu and Ding,2004)

There is no cl-complete left-c.e. real.

Theorem (Fan and Yu, 2012)

For any non-computable Δ^0_2 real α, there is a left-c.e. real β such that both of them have no common upper bound in left-c.e. reals under *cl*-reducibility

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The interplay between Turing and cl reducibility

Theorem (Barmpalias, Downey and Greenberg, 2010)

For a c.e. degree $\mathbf{d}$, the following are equivalent:

(1) $\mathbf{d}$ is array non-computable.

(2) There are left-c.e. reals $\alpha, \beta \in \mathbf{d}$ which have no common upper bound in the cl-degrees of left-c.e. reals.

(3) There is a left-c.e. real $\beta \in \mathbf{d}$ which is not cl-reducible to any random left-c.e. real.

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Definition

(α, β) ((A, B)) is a cl-maximal pair of left-c.e. reals (sets) if no left-c.e. real (c.e. set) can cl-compute both of them.

Theorem (Ambos-spies, Ding, Fan and Wolfgang, 2013)

For any c.e. set D the following are equivalent:

(1) $Deg_T(D)$ is array non-computable.

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Definition (Fan, 2016)

A c.e. Turing degree $\mathbf{d} = deg_T(D)$ is uniformly non-low_2 if for any total function $f \leq_T \emptyset'$ there is a uniform way to define a total function $g \leq_T D$ such that g is not dominated by f.

In the c.e. Turing degrees, $\{\mathrm{uniformly\ \ non-low_2}\} \subsetneq \{\mathrm{non-low_2}\} \subsetneq \{\mathrm{array\ \ non-computable}\}$.

Theorem (Fan, 2016)

If a c.e. Turing degree $\mathbf{d}$ is uniformly non-low_2, for any non-computable Δ^0_2 real α there is a left-c.e. real $\beta \in \mathbf{d}$ such that α and β have no common upper bound in left-c.e. reals under cl-reducibility.

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Theorem (Fan, 2016)

If a c.e. Turing degree $\mathbf{d}$ is uniformly non-low_2, for any non-computable Δ^0_2 real α there is a left-c.e. real $\beta \in \mathbf{d}$ such that α and β have no common upper bound in left-c.e. reals under cl-reducibility.

Density under cl-reducibility

Let standard reducibilities include S-reducibility, rK-reducibility, K-reducibility, and C-reducibility.

Proposition (Downey, Hirschfeldt, 2010)

Let r be a standard reducibility on the left-c.e. reals that is at least as strong as Solovay reducibility. Then the r-degrees of left-c.e. reals are dense.

Theorem (Barmpalias and Levis,2006)

The ibT-degrees of left-c.e. sets are not dense.

Theorem (Day, 2010)

The cl-degrees of left-c.e. sets are not dense.

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Theorem (Fan)

The *cl*-degrees of left-c.e. reals are not dense.

Proof sketch:

1. $P_e : \alpha \neq \Gamma_e^{\beta}$;
2. $R_e : \beta = \Delta_e^{\gamma_e} \wedge \gamma_e = \Phi_e^{\alpha} \Longrightarrow \exists \Theta[\alpha = \Theta^{\gamma_e}] \vee \exists \Psi[\gamma_e = \Psi^{\beta}]$.
3. To meet P_i, we build a diagonalization interval $I^i = [x_i, y_i]$.
4. Considering P_i, if I_e is unbounded for some $e \leq i$, how can we define an R_e-suitable diagonalization interval I^i?
5. Considering R_e, if I_e is unbounded, I, I' are two diagonalization intervals, how can we make sure both of them are consistent?

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Future work

Questions:

- For two left-c.e. real $\alpha <_{cl} \beta$, is there a d-c.e. real γ such that $\alpha <_{cl} \gamma <_{cl} \beta$?
- Is there a cl-maximal left-c.e. real?

Thank you!

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