Reverse mathematics and program termination

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Outline

1. Introduction
   - Program termination
   - Ramey's theorem
2. Termination theorem and Ramsey's theorem
   - Reverse mathematics
   - The strength of the termination theorem
   - Extracting information
3. Sharper analysis
   - Calculating bound

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Program

A program is a pair (I, R) where I is a set and $R \subseteq I^2$.

- I is a set of states. Each state $a \in I$ denotes the values of variables, e.g., $a = \langle x_0 = 3, x_1 = 2, y = 100 \rangle$.

  In this talk, we fix $I = \mathbb{N}$.
- R is said to be well-founded if there is no infinite sequence $\langle a_i \in I \mid i \in \omega \rangle$ such that $a_i R a_{i+1}$.
- Usually, R is generated by a computable transition function $\delta : I \to [I]^{<\omega}$ as $aRb \Leftrightarrow b \in \delta(a)$.
  (R is deterministic if $|\delta(a)| \leq 1$ for any $a \in I$.)
- A program is said to be terminating if R is well-founded.

To consider "termination" in this abstract setting, we just study well-foundedness of binary relations.

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Termination theorem

For given $R \subseteq \mathbb{N}^2$, we write $\mathrm{tcl}(R)$ for the transitive closure of R.

The following termination theorem is a basic tool of the study of program termination.

Theorem (Podelski/Rybalchenko)

For given $k \in \mathbb{N}$, we have the following.

> TT_k: *for any $R \subseteq \mathbb{N}^2$, R is well-founded if and only if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is well-founded.*

Termination theorem

$R \subseteq \mathbb{N}^2$ is said to be k-disjunctively well-founded if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is well-founded..

Theorem (Podelski/Rybalchenko)

For given $k \in \mathbb{N}$, we have the following.

> *A program terminates if and only if its transition relation is k-disjunctively well-founded.*

Example

Consider the following "program".

- variables: $x_0, \ldots, x_{n-1}$ (input), y (output).
- $x_0, \ldots, x_{n-1}$ are given by input, set $y := 0$.
- calculation:
  $\langle x_0, \ldots, x_{n-1}, y\rangle$
  $\Downarrow R$
  $\langle x_0 := x_0 + y, \ldots, x_{i-1} := x_{i-1} + y, x_i := x_i - 1, x_{i+1} := x_{i+1}, \ldots, x_{n-1} := x_{n-1}, y := y + 1\rangle$ for some $i = 0, \ldots, n-1$.
- Output y if $x_0 = \cdots = x_{n-1} = 0$.

Does this program terminate?

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- Output y if $x_0 = \cdots = x_{n-1} = 0$.

Does this program terminate?

Example

$\Rightarrow$ Yes!

- Define T_i ($i = 0, \ldots, n-1$) as
  $\langle x_0, \ldots, x_{n-1}, y \rangle T_i \langle x'_0, \ldots, x'_{n-1}, y' \rangle$
  $\iff x'_i < x_i$.
- Then, $\mathrm{tcl}(R) = T_0 \cup \cdots \cup T_{n-1}$, and each T_i is well-founded.
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Termination theorem

One can apply the termination theorem for functional programs.

Theorem (Kuwahara/Terauchi/Unno/Kobayashi)

For given $k \in \mathbb{N}$, we have the following.

> *A functional program terminates if and only if its call relation is k-disjunctively well-founded.*

Ramsey's theorem (for pairs)

Termination theorems are proved by using Ramsey's theorem.

Definition (Ramsey's theorem)

- Ramsey's theorem (RT^2_k): for any $P : [\mathbb{N}]^2 \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
- Weak Ramsey's theorem (WRT^2_k): for any coloring $P : [\mathbb{N}]^2 \to k$, there exists $H = \{h_0 < h_1 < \dots\}$ such that for any $i, j \in \mathbb{N}$, $P(h_i, h_{i+1}) = P(h_j, h_{j+1})$.

Here, $[\mathbb{N}]^2 = \{(x, y) \in \mathbb{N}^2 \mid x < y\}$.

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Ramsey's theorem (for pairs)

Definition (Ramsey's theorem)

- Ramsey's theorem (RT^2_2): every infinite complete graph with its edges colored by red or blue has an infinite complete monochromatic subgraph.
- Weak Ramsey's theorem (WRT^2_2): every infinite complete graph with its edges colored by red or blue has an infinite monochromatic path with the increasing order.

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Proof-theoretic analysis for termination theorem

Project by Stefano Berardi, Paulo Oliva and Silvia Steila

> "Obtain a priori-bounds for the termination of computer programs, and compare these with bounds obtained via other intuitionistic proofs of the Termination Theorem."

Roughly speaking,

- if we would know the strength of the Termination theorem in some proof-theoretic settings, e.g., constructive mathematics or reverse mathematics, one would extract some information of a bound of termination from the proof.

⇒ Try to measure the strength of the termination theorem (reverse mathematics)!

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What is Reverse Mathematics?

Hilbert's reductionism program (1920s):

Find a good axiomatic system T for the entire mathematics, and prove the 'consistency of T' by a 'finitistic method'.

- This program failed because of Gödel's incompleteness theorem (1930).

$\Rightarrow$ Which axioms are exactly needed for mathematics?

$\Rightarrow$ **Reverse Mathematics**

H. Friedman's theme (1976):

very often, if a theorem τ of ordinary mathematics is proved from the "right" axioms, then τ is equivalent to those axioms over some weaker system in which itself is not provable.

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What is Reverse Mathematics?

Reverse Mathematics program (Friedman Simpson program)

1. Formalize the theorem τ of "core of math" within an appropriate axiomatic system.
2. Find the weakest axiom T in which we can prove τ.
3. Classify "core of math" using the logical strength.

   ("core of math": basic theorems of analysis, algebra, geometry, etc.)

Reverse mathematics

First, we fix the base system RCA_0.
It consists of the following.

Basic axioms for $+, \times$, exp, $<, \ldots$

Any computably definable (with an oracle) sets exist.

Any function defined by primitive recursion is total.

Over RCA_0, we compare mathematical theorems, with the following so called "big five" axioms.

- WKL: Weak König's lemma (for binary trees only).
- KL: König's lemma.
- ATR: Comparability of well-orderings.

$\vdots$

Reverse mathematics over RCA_0

The following are provable within the base system RCA_0.

- Intermediate value theorem / mean value theorem.
- Fundamental theorem of algebra.

The following are equivalent to WKL over RCA_0.

- Compactness of closed bounded intervals on $\mathbb{R}$.
- Every countable ring has a prime ideal.
- Jordan curve theorem.

The following are equivalent to KL over RCA_0.

- Every countable ring has a maximal ideal.
- Every countable vector space has a basis.

However, RT^2_2 or WRT^2_2 is not equivalent to any of the above.
In fact, $\mathsf{RCA}_0 < \mathrm{WRT}^2_2 < \mathrm{RT}^2_2 < \mathrm{KL}$ and $\mathrm{WRT}^2_2, \mathrm{RT}^2_2 \ngeq \mathrm{WKL}$.

Reverse Mathematical result

We do reverse mathematics for the termination theorem.

Theorem

The following are equivalent over RCA_0.

1. WRT^2_k.
2. TT_k: *for any* $R \subseteq \mathbb{N}^2$, *R is well-founded if and only if there exists* $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ *such that* $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ *and each* T_i *is well-founded.*

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What about RT^2_2?
Can we strengthen the termination theorem?

Theorem (Berardi/Steila)

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Here, $R \subseteq \mathbb{N}^2$ is said to be *H*-well-founded if there is no infinite chain $\langle a_0, a_1, \ldots \rangle$ such that $a_i R a_j$ if $i < j$.

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Extract some information from RM

Corollary

The following is provable within $\mathrm{RCA}_0 + \mathrm{WRT}^2_2$ *for each* $k \in \omega$.

> *A program terminates if and only if its transition relation is* k*-disjunctively well-founded.*

Now we want extract some information from this.
We will see some limitation of a termination verifier based on the above.

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Idea.

- Assume that the soundness of a termination verifier V is proved in a system T.
- If V says that a program P terminates, then T proves the termination of P.
- "a program P terminates" is a Π_2-statement.
- Thus, if we know the Π_2-consequences of T, then we know the limitation of the verification power of V.

Note that this kind of method is used (based on first-order arithmetic) in the study of termination of term rewriting, e.g., by Bucholz (1995).

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By these four results in proof theory/reverse mathematics,

- Parsons(1970): if $\mathrm{I}\Sigma_1$ proves a Π_2-statement $\forall n \exists k \psi(n,k)$, then there exists a primitive recursive function f such that $\forall n \exists k \leq f(n) \psi(n,k)$ holds.
- Kirby/Paris(1977): $\mathrm{B}\Sigma_2$ is Π_3-conservative over $\mathrm{I}\Sigma_1$.
- Chong/Slaman/Yang(2012): $\mathsf{RCA}_0 + \mathrm{CAC}$ is conservative over $\mathrm{B}\Sigma_2$.
- Murakami/Yamazaki/Y(2014): $\mathsf{RCA}_0 + \mathrm{CAC}$ proves WRT^2_2.

we have the following.

Theorem

If $\mathsf{RCA}_0 + \mathrm{WRT}^2_2$ proves a Π_2-statement $\forall n \exists k \psi(n,k)$, then there exists a primitive recursive function f such that $\forall n \exists k \leq f(n) \psi(n,k)$ holds.

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Thus, we have the following.

Corollary

If the termination of a program P is verified by the P/R termination theorem, then there exists a primitive recursive function f such that for any input n for P, P terminates in $f(n)$ steps.

Thus, the termination of the Ackermann function cannot be verified by this method.

⋆ The above can be shown by many different methods, e.g., using the analysis of Dickson's lemma (Figueira/Figueira/Schmitz).

The exact strength of Ramsey's theorem for pairs

Recently, it is shown that the strength of (full) Ramsey's theorem for pairs is still as same as primitive recursive functions.

Theorem (Patey Y, 2015)

If $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *proves a* Π_2*-statement* $\forall n \exists k \psi(n,k)$*, then there exists a primitive recursive function* f *such that* $\forall n \exists k \leq f(n) \psi(n,k)$ *holds.*

Outline

Termination theorem with bound

$f : \mathbb{N} \to \mathbb{N}$ is said to be a bound of R if any R-sequence starting from a is shorter than $f(a)$.
R is said to be bounded if it has a bound.

Theorem

For given $k \in \mathbb{N}$, we have the following.

$\mathrm{TT}^{bd}_{k,f}$: *for any $R \subseteq \mathbb{N}^2$, R is bounded if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is bounded by f.*

Then, can we calculate a bound for R by f?

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Paris-Harrington theorem (for pairs)

Definition (Paris-Harrington theorem)

- Paris-Harrington theorem (PH^2_k): for any $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $P : [[a,b]]^2 \to k$, there exists a set $H \subseteq [a,b]$ such that $|P([H]^2)| = 1$ and $|H| > \min H$.
- Weak Paris-Harrington theorem (WPH^2_k): for any $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $P : [[a,b]]^2 \to k$, there exists $H = \{h_0 < h_1 < \cdots < h_m\} \subseteq [a,b]$ such that for any $i,j < m$, $P(h_i, h_{i+1}) = P(h_j, h_{j+1})$ and $|H| > \min H$.

Define

- $H_k(a) = \min\{b \mid [a,b] \text{ enjoys the condition for } \mathrm{PH}^2_k\}$.
- $W_k(a) = \min\{b \mid [a,b] \text{ enjoys the condition for } \mathrm{WPH}^2_k\}$.

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- $\mathrm{WPH}^2_{k,f}$: for any $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $P : [[a,b]]^2 \to k$, there exists $H = \{h_0 < h_1 < \cdots < h_m\} \subseteq [a,b]$ such that for any $i, j < m$, $P(h_i, h_{i+1}) = P(h_j, h_{j+1})$ and $|H| > f(\min H)$.

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- $W^f_k(a) = \min\{b \mid [a,b]$ enjoys the condition for $\mathrm{WPH}^{2,f}_k\}$.

bdd-Termination vs PH

Theorem (WKL_0)

For any $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ the following are equivalent.

1. $\mathrm{WPH}^2_{k,f}$.
2. $\mathrm{TT}^{bd}_{k,f}$.

More precisely, for 1 → 2, if $\mathrm{tcl}(R)$ is k-disjunctively bounded by f, then R is bounded by W^f_k.

Note that if f is primitive recursive and k is standard, then $\mathrm{WPH}^2_{k,f}$ is provable within RCA_0. Thus, W^f_k is bounded by a primitive recursive function (by Person's theorem).

Corollary

R has a primitive recursive bound if and only if R is k-disjunctive primitive recursively bounded.

Fast growing functions

Let F_k be the usual k-th fast growing function defined as

$$\begin{cases} F_0(x) = x + 1, \\ F_{n+1}(x) = F_n{}^{(x+1)}(x). \end{cases}$$

Theorem (Ketonen/Solovay)

- $W_k \leq H_k \leq F_{k+4}$.
  More generally, $W_k^{F_n} \leq H_k^{F_n} \leq F_{k+n+4}$.

Theorem (from recent termination analysis)

- $W_k \leq F_{k+1}$.
  More generally, $W_k^{F_n} \leq F_{k+n+1}$.

Termination theorem with bound

Theorem

For given $k \in \mathbb{N}$, we have the following.

$\mathrm{TT}_k^{F_n}$: *for any $R \subseteq \mathbb{N}^2$, R is bounded by F_{k+n+1} if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is bounded by F_n.*

This upper bound is sharp.

Fact

The program for F_k is k-disjunctive linearly (F_0-) bounded.

Termination theorem with bound

Theorem

For given $k \in \mathbb{N}$, we have the following.

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This upper bound is sharp.

Fact

The program for F_k is k-disjunctive linearly (F_0-) bounded.

Termination theorem with bound (sharper version)

If R is a relation for a deterministic program, we have the converse.

Theorem

For given $k \in \mathbb{N}$, we have the following.

> *for any deterministic program $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $T_0, \ldots, T_{k+1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k+1}$ and each T_i is bounded by F_0.*

Corollary

R has a primitive recursive bound if and only if R has k-disjunctive linearly bounded for some k if and only if R has k-disjunctive linearly H-bounded for some k.

Termination theorem with bound

Question

Can we do a similar analysis for disjunctively well-roundedness of call relations?

Conclusion

- Termination theorem for programs is equivalent to Ramsey's theorem for pairs.
- A bound for a terminating program can be given by the Paris/Harrington number, and it is almost optimal.
- In general, proof-theoretic method would be applied to the study of termination.

Thank you for listening!

Conclusion

- Termination theorem for programs is equivalent to Ramsey's theorem for pairs.
- A bound for a terminating program can be given by the Paris/Harrington number, and it is almost optimal.
- In general, proof-theoretic method would be applied to the study of termination.

Thank you for listening!

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